Computation of (*): The normalisation of

$K(s,x,y)$ is such that

$$\lim_{s\to 0} K(s,x,y) = \lim_{s\to\infty} \langle x| e^{is\not{D}^2} |y\rangle = \langle x|y\rangle = \delta^4(x-y)$$

This leads to the Ansatz:

$$K(s,x,y) = \frac{1}{2\pi}\frac{i}{s} e^{i\frac{|x-y|}{4s}} \left(1+O(s)\right)$$

In addition $K(s,x,y)$ should solve the diff. equation:

$$\frac{1}{i}\partial_s K(s,x,y) = \not{D}_x^2 \, K(s,x,y) \text{. Then, using}$$

$$\not{D}^2 = \gamma^\mu D_\mu \gamma^\nu D_\nu = \frac{1}{4}[\gamma^\mu,\gamma^\nu][D_\mu,D_\nu] + \underbrace{g^{\mu\nu} D_\mu D_\nu}_{\Box}$$

We can expand

$$e^{is\not{D}^2} \sim e^{is\Box}\Big(1 + \frac{si}{4}[\gamma^\mu,\gamma^\nu]\overbrace{[D_\mu,D_\nu]}^{iF_{\mu\nu}} + O(s^2)\Big)$$

we find

$$K(s,x,x) \sim \frac{i}{2\pi}\frac{1}{s}\left(1 + \frac{is}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu} + O(s^2)\right)$$

Thus,

$$\operatorname{tr}\left(e^{is\not{D}^2}\right) \sim \frac{i}{2\pi}\frac{1}{s}\int d^2x - \frac{1}{8\pi}\int[\gamma^\mu,\gamma^\nu]F_{\mu\nu}\,d^2x + O(s)$$

this term cancels betwee $H_{\varphi\varphi^*}$ and $H_{\bar\varphi\bar\varphi^*}$

Quantum Effective action:

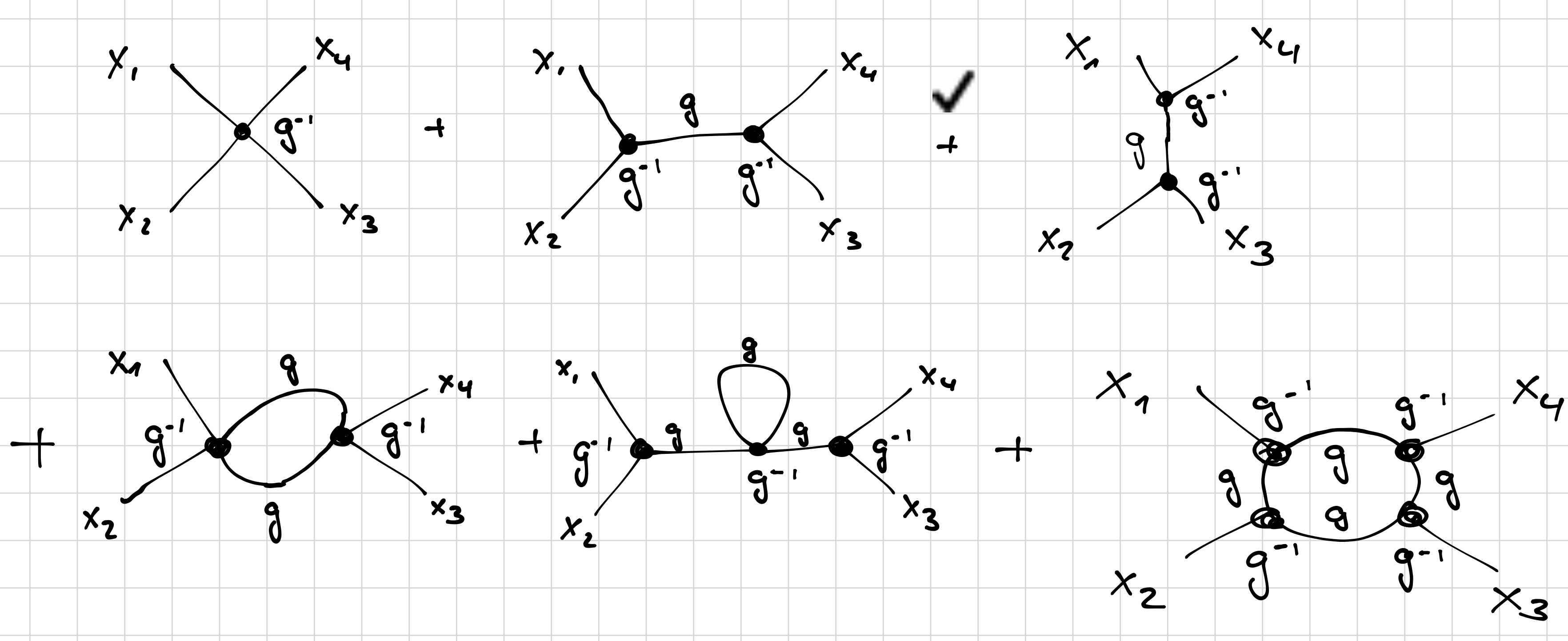
All connected correlation functions are uniquely determined by the Schwinger functional

$$e^{\frac{i}{\hbar} W[J]} = \int D[\phi]\, e^{\frac{i}{\hbar} S' + i \int J \phi} \Big|_{\bar{\phi} = \frac{\delta 4}{\delta \phi}}$$

Therefore, $W[J]$ encodes the complete information of the quantum theory. For instance, the connected 4-pt correlation function is given by

$$\langle \phi(x_1) \cdots \phi(x_4) \rangle = \frac{\delta^4 W[J]}{\delta J(x_4) \cdots \delta J(x_1)}$$

with (Feynman) graphical representation: e.g. $S = \frac{1}{g} \int (\partial\phi)^2 + \phi^3 + \phi^4$



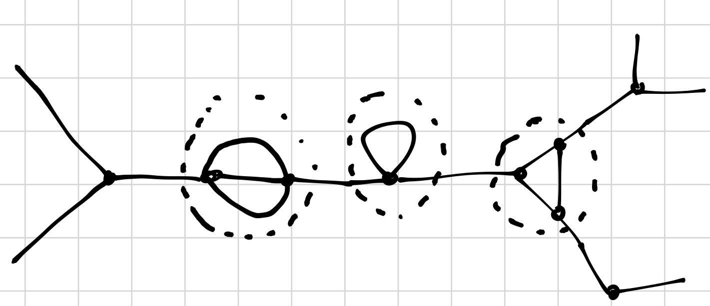
$+ \ldots$

The *tree-level* diagramms in the first line $O(g^{-1})$ are correctly reproduced by the stationnary phase approximation of $W[J]$ that corresponds to solving the e.m. for $S[\phi]$ with sources $J(x)$. These are thus classical contributions that don't involve quantum fluctuations around the classical path. These contributions don't lead to infinities provided the current $J(x)$ is chosen appropriately.

The one-loop diagramm $O(g^0)$ in the second line, on the other hand originate from quantum fluctuations around the "classical path" and involve products of Green functions (= distributions) at identical points and are thus infinite unless they are suitably regularised.

In a renormaliseable theory, the divergences that arise when removing the regulator can be absorbed in an (infinite) renormalisation of the fields, the masses and the coupling constant, g of the classical action. For instance in a scalar phi^4 theory the first and the second diagram in the second line contribute to renormalisation of coupling constant and mass respectively.

For now let us just note that, in order to understand the issues of renormalisation, it is sufficient to focus on the loops in the respective diagramms. For instance, for the diagram in



the relevant information is already contained in the sub diagramms contained in the dotted circles. The global structure of the diagram in provides no new information for renormalisation since the «tree-level « composition of these sub-diagrammms gives no further infinities. As such the Schwinger functional contains redundant information as far as renormalisation is concerned.

There is a generating functional Γ for just these type of proper diagram given by the Legendre transform of the Schwinger functional,

$$\Gamma(\underline{\Phi}) \equiv W[J] - \int d^4x \, \underline{\Phi}^I J_I$$

here, J is implicitly determined by

$$\underline{\Phi}^I = \frac{\delta W[J]}{\delta J_I}$$

in analogy with p and q in Lagrangian mechanics, Analogously,

$$\frac{\delta \Gamma(\underline{\Phi})}{\delta \underline{\Phi}^S} = - J_S$$

$\Gamma[\Phi]$ has an expansion in Φ as

$$\Gamma[\Phi] = \frac{1}{2}\iint \Gamma^{(2)}(x,y)\, \Phi(x)\Phi(y)$$

$$+ \frac{1}{3!}\iiint \Gamma^{(3)}(x,y,z)\, \Phi(x)\Phi(y)\Phi(z)$$

$$+ O(\Phi^4)$$

$\therefore$ $\Gamma^{(n)}$ can involve derivatives (pseudo diff. ops)

As a warm-up let us compute $\Gamma^{(2)}$ at the classical level. If we write

$$S[\Phi] = \frac{1}{2}\int (\partial\phi)^2 - m^2\phi \; + S_{int}$$

then S_{int} does not contribute to $\Gamma^{(2)}$ at the classical level since S_{int} contributes terms like

but not to

at tree level. We then have

$$e^{\frac{i}{\hbar}W[J]} = e^{\frac{i}{\hbar}S[\phi_{cl}] + \frac{i}{\hbar}\int J\phi_c} \quad ; \quad -(\Box+m^2)\phi_{cl} + J = 0$$

saddle pt approx

$$\Rightarrow W[J] = \frac{1}{2}\int J\,\frac{1}{\Box+m^2}\,J$$

$$\Gamma^{(2)}[\Phi] = \frac{1}{2}\iint \Gamma^{(2)}(x,y)\,\Phi(x)\Phi(y) = W[J] - \int J\Phi\,\Big|_{\Phi = \frac{\delta W}{\delta J}}$$

$$\Phi(x) = \frac{\delta}{\delta J(x)}\,\frac{1}{2}\int J(y)\langle y|\frac{1}{\Box+m^2}|\tilde{y}\rangle J(\tilde{y}) = \left(\frac{1}{\Box+m^2}J\right)(x)$$

Then

$$\Gamma^{(2)}[\Phi] = -\frac{1}{2}\int \Phi\,(\Box^2+m^2)\,\Phi = S^{(2)}[\Phi] .$$