(https://en.m.wikipedia.org/wiki/Fujikawa_method)

Consider the fermionic path-integral:

$$\not{D}\psi_i = -\lambda_i \psi_i .$$

The eigenfunctions are taken to be orthonormal with respect to integration in d-dimensional space,

$$\delta_i^j = \int \frac{d^d x}{(2\pi)^d} \psi^{\dagger j}(x)\psi_i(x).$$

The measure of the path integral is then defined to be:

$$\mathcal{D}\psi\mathcal{D}\overline{\psi} = \prod_i da^i db^i$$

Under an infinitesimal chiral transformation, write

$$\psi \to \psi' = (1 + i\alpha\gamma_{d+1})\psi = \sum_i \psi_i a'^i,$$

$$\overline{\psi} \to \overline{\psi}' = \overline{\psi}(1 + i\alpha\gamma_{d+1}) = \sum_i \psi_i b'^i.$$

The Jacobian of the transformation can now be calculated, using the orthonormality of the eigenvectors

$$C_j^i \equiv \left(\frac{\delta a}{\delta a'}\right)_j^i = \int d^d x\, \psi^{\dagger i}(x)[1 - i\alpha(x)\gamma_{d+1}]\psi_j(x) = \delta_j^i - i\int d^d x\, \alpha(x)\psi^{\dagger i}(x)\gamma_{d+1}\psi_j(x).$$

The transformation of the coefficients $\{b_i\}$ are calculated in the same manner. Finally, the quantum measure changes as

$$\mathcal{D}\psi\mathcal{D}\overline{\psi} = \prod_i da^i db^i = \prod_i da'^i db'^i \det{}^{-2}(C_j^i),$$

where the Jacobian is the reciprocal of the determinant because the integration variables are Grassmannian, and the 2 appears because the a's and b's contribute equally. We can calculate the determinant by standard techniques:

$$\begin{aligned}\det{}^{-2}(C_j^i) &= \exp\left[-2\mathrm{tr}\ln(\delta_j^i - i\int d^d x\, \alpha(x)\psi^{\dagger i}(x)\gamma_{d+1}\psi_j(x))\right] \\ &= \exp\left[2i\int d^d x\, \alpha(x)\psi^{\dagger i}(x)\gamma_{d+1}\psi_i(x)\right]\end{aligned}$$

to first order in α(x).

Specialising to the case where α is a constant, the Jacobian must be regularised because the integral is ill-defined as written. Fujikawa employed heat-kernel regularization, such that

$$\begin{aligned}-2\mathrm{tr}\ln C_j^i &= 2i\lim_{M\to\infty}\alpha\int d^d x\, \psi^{\dagger i}(x)\gamma_{d+1}e^{-\lambda_i^2/M^2}\psi_i(x) \\ &= 2i\lim_{M\to\infty}\alpha\int d^d x\, \psi^{\dagger i}(x)\gamma_{d+1}e^{\not{D}^2/M^2}\psi_i(x)\end{aligned}$$

The final expression then reproduces $(\Delta S')_{reg}$ .To summarise, the quantum BV equation is (a generalisation of) a consistency condition for the ST identities.