Next we want to develop an intuition about the Δ-operator. For this we consider as example chiral, massless QED in 1+1 dimensions, with BV action

$$\Psi = \begin{pmatrix} \chi \\ \varphi \end{pmatrix} \quad : \quad \not{D}\Psi = \gamma^\mu(\partial_\mu + iA_\mu)\Psi$$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} = 2\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$\gamma^0 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = i\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\not{D}\Psi = \begin{pmatrix} 0 & \partial_t - \partial_x + i(A_t - A_x) \\ \partial_t + \partial_x + i(A_t + A_x) & 0 \end{pmatrix}\begin{pmatrix} \chi \\ \varphi \end{pmatrix}$$

chiral: $\chi = 0$: $\not{D}\Psi = (\partial_- + iA_-)\varphi$

$\bar{\varphi}$: $\left[\overset{\text{Dirac}}{\bar{\Psi}} = \Psi^{cc}\gamma^0 = \begin{pmatrix} \chi^{cc} \\ \varphi^{cc} \end{pmatrix} \equiv \begin{pmatrix} \bar{\varphi} \\ \bar{\chi} \end{pmatrix}\right]$

$$S_{BV} = \underbrace{\int -\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + i\bar{\varphi}(\partial_- + iA_-)\varphi}_{S_0[F_{\mu\nu}, \varphi]}$$

$$+ \int \underbrace{\partial_\mu H}_{\delta(A_\mu)} A^{*\mu} \underbrace{- iH}_{\delta(\varphi)}\varphi\varphi^* \underbrace{+ iH}_{\delta(\bar{\varphi})}\bar{\varphi}\bar{\varphi}^*$$

$$+ \underbrace{\int \bar{H}^* b}_{\text{trivial pair}}$$

BV-BRST transformation:

$$\delta\varphi^* = \delta\xi\,(S,\varphi^*) = \delta\xi\,\frac{\delta_R S}{\delta\varphi} = -\delta\xi\,(iD_-\bar\varphi + iH\varphi^*)$$

$$\delta\bar\varphi^* = \delta\xi\,(S,\bar\varphi^*) = \delta\xi\,\frac{\delta_R S}{\delta\bar\varphi} = -\delta\xi\,(iD_-\varphi - iH\bar\varphi^*)$$

$$\delta A_\mu^* = \delta\xi\,(S,A_\mu) = \delta\xi\,\frac{\delta_R S}{\delta A_\mu} = -\delta\xi\left(\frac{1}{e^2}\partial_\nu F_\mu{}^\nu + \delta_{\mu-}\bar\varphi\varphi\right)$$

$$\delta A_\mu = \delta\xi\,(S,A_\mu) = -\delta\xi\,\frac{\delta_R S}{\delta A_\mu^*} = -\delta\xi\,\partial_\mu H$$

$$\delta\varphi = \delta\xi\,(S,\varphi) = -\delta\xi\,\frac{\delta_R S}{\delta\varphi^*} = i\delta\xi\,H\varphi$$

$$\delta\bar\varphi = \delta\xi\,(S,\bar\varphi) = -\delta\xi\,\frac{\delta_R S}{\delta\bar\varphi^*} = -i\delta\xi\,H\bar\varphi$$

$$(S,S) = \delta_\xi S = -\frac{1}{2e^2}\int F_{\mu\nu}\overbrace{\delta_\xi F^{\mu\nu}}^{=0}$$

$$+\int \underbrace{i\bar\varphi D_-(i\delta\xi H\varphi) + i(-i\delta\xi H\bar\varphi)D_-\varphi + i\bar\varphi(-i\delta\xi\,\partial_- H)\varphi}_{\delta_\xi(i\bar\varphi D_-\varphi)\ =\ (*)\ \equiv\ 0}$$

$$+\frac{1}{e^2}\int \partial_\mu H\left(\cancelto{0}{-\delta\xi\,\partial_\nu F_\mu{}^\nu}\right)$$

$$\overbrace{+\int \partial_- H\,(-\delta\xi\,\underbrace{\bar\varphi\varphi}_{①}) - iH\varphi\underbrace{(-\delta\xi\, iD_-\bar\varphi)}_{②} + iH\bar\varphi\underbrace{(-\delta\xi\, iD_-\varphi)}_{③}}^{=0 \text{ by partial integration}}$$

composite operators in QFT

need to be regularised...

Regularisation of S': e.g point splitting, $S' \to S_\varepsilon$ by the substitutions:

$$\bar\varphi\, \varphi \to \bar\varphi(x-\varepsilon)\, e^{-i\int_{x-\varepsilon}^{x+\varepsilon} A_\mu dy^\mu} \varphi(x+\varepsilon)$$

$$\bar\varphi\, D_-\varphi \to \lim_{\varepsilon\to 0} \bar\varphi(x-\varepsilon)\, e^{-i\int_{x-\varepsilon}^{x+\varepsilon} A_\mu dy^\mu} D_-\varphi(x+\varepsilon)$$

etc.

We then define $(S', S')_{reg} = \lim_{\varepsilon\to 0} (S'_\varepsilon, S'_\varepsilon)$

Then ①, ② and ③ cancel against each other. Thus $(S', S')_{reg}$ is well defined (and zero) as an operator. Let us now turn to ΔS: as a distribution we have

$$\Delta S' = \int \frac{\delta_R \delta_L S}{\delta \bar\Phi^I \delta \bar\Phi^{*I}} = \int \underset{\frac{\delta^2}{\delta\varphi\delta\varphi^*}}{H\, \delta(0)} - \int \underset{\frac{\delta^2}{\delta\bar\varphi\delta\bar\varphi^*}}{H\, \delta(0)}$$

which is ill-defined. As an operator we have similarly, $H\, \mathrm{tr}(\mathbb{1}) - H\, \mathrm{tr}(\mathbb{1})$

where the trace is over $\mathcal{H}_{\varphi\varphi^*}$ and $\mathcal{H}_{\bar\varphi\bar\varphi^*}$ respectively. One way to regularise $\Delta S'$ is to define

$$\Delta_S \equiv \int d^4z\, d^4z' \left(\underset{\text{Fermion}}{\frac{\delta}{\delta \bar\Phi^{*I}(z)}} K^F_S(z,z') \underset{\text{Fermion}}{\frac{\delta}{\delta \bar\Phi^I(z')}} + \frac{\delta}{\delta \bar A_\mu(z)} K^B_S(z,z') \frac{\delta}{\delta A_\mu(z')} \right)$$

where $\begin{cases} K_s^F(z,z') = \langle z | e^{is\not{D}^2} | z' \rangle \\ K_s^B(z,z') = \langle z | e^{is\Box} | z' \rangle \end{cases}$

Then $$\Delta\left(e^{\frac{i}{\hbar}S}\right)_{reg} := \lim_{s\to 0}\lim_{\varepsilon\to 0} \Delta_s \left(e^{\frac{i}{\hbar}S'_\varepsilon}\right)$$

Exercise $$= -\frac{1}{2\hbar^2}(S'_\varepsilon, S'_\varepsilon)_s + \frac{i}{\hbar}\Delta_s S'_\varepsilon$$

Then, using the (heat-kernel) asymptotic expansion
$$tr_{diff}\langle x | e^{is\not{D}^2} | y \rangle \sim \# \frac{1}{s}\left(1 + s\, \sigma\gamma^{\mu\nu}F_{\mu\nu} + O(s^2)\right)$$
we find that
$$(\Delta S')_{reg} = 2\# F_{+-} \neq 0$$
where the factor of 2 comes from the fact that φ and φ^* are oppositely charged. Thus the quantum BV-equation is not satisfied. This theory is anomalous.

Let us finally argue that $(\Delta S)_{reg} \neq 0$ is related to the non-invariance of the measure. The following extract is from Wikipedia