# 1) Construction of gauge theories

**Rep:** Noether charge in a Lorentz covariant theory

$$Q = \int d^3x \; j_0(\underline{x}, t) \; ; \; \partial_\mu j^\mu(x) = 0$$

$$L_{int} = A_\mu(x) j^\mu(x) \sim (A_\mu + \partial_\mu \lambda) j^\mu$$

interaction

$\longrightarrow$ $A_\mu(\underline{x}, t)$ is partly redundant.

vector pot.

matter

QED: $\mathcal{L}_{int} = A_\mu(x)\, j^\mu(x)$ (source) ✓

$\boxed{\mathcal{L}_{matter} = ?}$

$\mathcal{L}_{gauge} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ & 0 & B_3 & \cdot \\ & & \cdot & \cdot \end{pmatrix}$$

$\mathcal{L}_g + \mathcal{L}_{int} \xrightarrow{E.L.}$ Maxwell equ's $\partial_\mu F^{\mu\nu} = 4\pi j^\nu$ ✓

$$j^\mu(x) = \bar{\Psi}(x)\gamma^\mu \Psi(x)$$

$$\Psi_A(x) = \begin{pmatrix} \phi_\alpha(x) \\ \chi^{\dot\alpha}(x) \end{pmatrix}$$

Weyl spinors

$\alpha = 1, 2$.

$\dot\alpha = 1, 2$.

$A = 1, \dots, 4$

$\phi_\alpha$ in a representation $SL(2,\mathbb{R})$

quant. mech. Lorentz gr.

$\therefore$ in non-rel. QM $SL(2,\mathbb{R}) \rightsquigarrow SU(2)$

QM rot. group

$\bar{\Psi} = \Psi^\dagger \gamma^0$ ; $\gamma^\mu$ : $\gamma$-matrices

$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

Mink metric

$(-,+++)$

$$\boxed{j^\mu = \bar{\Psi}\gamma^\mu\Psi}$$

matter field

Symm: $\Psi(x) \longmapsto e^{-i\theta(x)}\Psi(x)$

local (U(1)) invariance.

$$\mathcal{L}_m = \bar{\Psi} e^{+i\theta} \big(i\gamma^\mu \partial_\mu - \overset{\text{mass}}{m}\big) e^{-i\theta(x)} \Psi$$

$$\boxed{\not{\partial} \equiv \gamma^\mu \partial_\mu} \qquad \xrightarrow{E.L.} (i\not{\partial} - m)\Psi = 0$$

$\longrightarrow$ Not invariant!

$$\bar{\Psi}\, i\not{\partial}\, \Psi \longrightarrow \bar{\Psi}\, i\gamma^\mu(\partial_\mu + ieA_\mu)\Psi \quad \sim \underbrace{e A_\mu j^\mu}_{\mathcal{L}_{int}}$$

Inv. $\quad \Psi \to e^{-i\theta(x)}\Psi ; \quad A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \theta$

Action:

$$I_{QED}(\psi, \bar\psi, A_\mu) =$$

renorm

$$\int d^4x \left\{ \underline{i \bar\psi (\not{D} + im) \psi} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right\}$$

dictated by g.i.

$$\not{D} = \gamma^\mu D_\mu \, , \quad D_\mu = \partial_\mu + ieA_\mu$$

Rem: gauge inv. does not fix $I_{QED}$ completely e.g. $\Delta I_{n.m.} = \int d^4x \, \bar\psi [\gamma^\mu, \gamma^\nu] \psi F_{\mu\nu}$ not r.e.

(Exercise: check g.i.) not

$$I = I_{QED} + \Delta I_{n.m.}$$

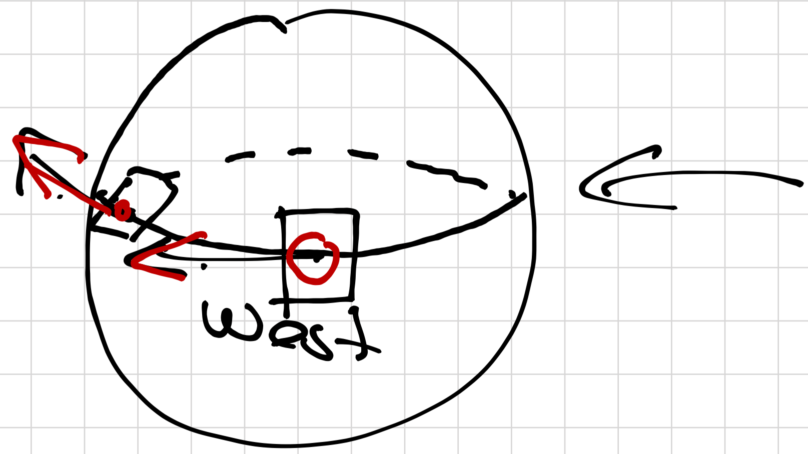
Interpretation: $A_\mu(x)$ (connection) defines what is a constant field (section):

i) $A_\mu \equiv 0$ : $\partial_\mu \psi = 0$ : $\psi = \text{const.}$

$A_\mu = \frac{1}{e}\partial_\mu \Theta(x)$ : $(\partial_\mu + ieA_\mu)\psi = 0$

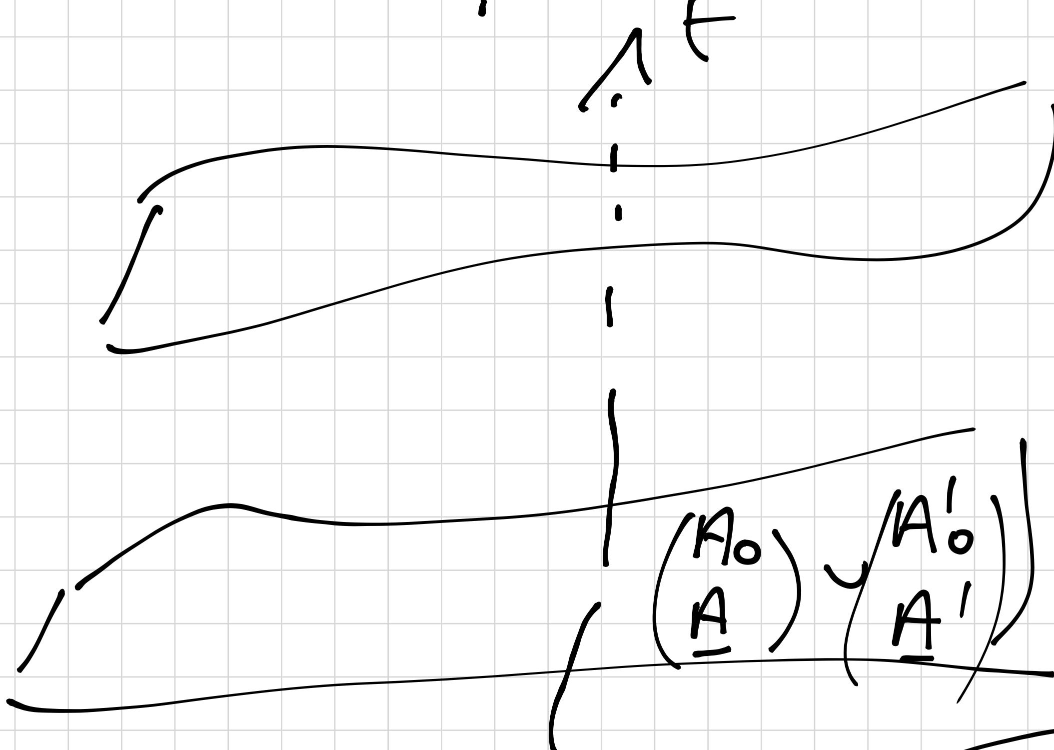
$\psi = e^{-i\Theta(x)} 1 = \text{const.}$

ii) Probl. sh. 1: QED




gauge principle: The choice of frame should be a local choice.

Hamiltonian form:




$$\underline{A}'(\underline{x},t) = \underline{A}(x,t) + \underline{\nabla}\theta(\underline{x})$$

# II) Non-abelian gauge theory

- $A_\mu$ is replaced by a Lie-algebra valued vector pot.

"approximation" to a Lie group

Rep: phase:
$$e^{-i\theta_2(x)} \cdot e^{-i\theta_1(x)} = e^{-i(\theta_1+\theta_2)(x)}$$

→ continuous group with several parameters

Noether charges

generalisation of phase inv.

Example: rotation SO(3)

$$SO(3) \ni U = \overbrace{R(e_z,\alpha)\circ R(e_x,\beta)\circ R(e_y,\gamma)}^{\text{Euler angles}}$$

$$\neq R(e_x,\beta)\circ R(e_z,\alpha)\circ R(e_y,\gamma)$$

SO(3) is a Lie group

Lie algebra: $R(e_z,\alpha) = e^{i\alpha K_3}$

$$K_3 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

generators

$R(e_x,\beta) = e^{i\beta K_1}$

$\boxed{[K_1, K_2] = iK_3}$ e.t.c

$K_1, K_2, K_3$ form a basis of the Lie algebra $so(3)$

Near $\mathbb{1}$ $\underbrace{so(3)}_{\text{Lie alg.}}$ determines $\underbrace{SO(3)}_{\text{Lie group}}$

More generally, a <u>Lie group</u> $G$ is obtained by "exponentiating" a <u>Lie algebra</u> $\mathfrak{g}$

$\mathfrak{g}$ contains almost all of the relevant information.

General def'n of Lie algebras:

A Lie algebra is a vector space $\mathfrak{g}$ with a binary operation

(1) $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$

$z, y, x \in \mathfrak{g}$, (2) $[x, y] = -[y, x]$

(3) $[ax + by, z] = a[x, z] + b[y, z]$

$a, b \in \mathbb{C}$

(4) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$

Jacobi identity

If $\{T_a\}$ are the generators of $\mathfrak{g}$

then

$$[T_a, T_b] = i \underbrace{f_{ab}{}^{c}}_{-f_{ba}{}^{c}} T_c$$

(summe over c undrs)

structure constant

characterize the algebra (group)

$C_{ab}{}^{c} = 0 \quad \forall a,b,c$

$\Rightarrow$ abelian group

In physics $\{T_a\}$ typically are the generators of some fundamental representation of $\mathfrak{g}$ ($G$)

$$i C_{ab}{}^{c} \equiv \left(T_a^{\mathrm{adj}}\right)^{c}{}_{b} \quad \text{form}$$

an adjoint representation

cf. sheet 2

- $A_\mu$ Lie algebra valued means

$$A_\mu(x) = \underbrace{A^a_\mu(x)}_{\text{coeff}} \underbrace{T_a}_{\text{gener.}} \in \mathfrak{g}$$

- The matter fields (e.g. quarks) transform in a fundamental rep.

i.e

$$\underset{\substack{\uparrow \\ \text{spinor}}}{\psi_A(x)} \xmapsto{g.t.} \psi^{(u)}_A(x) = \underbrace{U(x)}_{\substack{= e^{-i\theta^a T_a} \\ \uparrow \\ \text{near } \mathbb{1}}} \psi_A(x)$$

Example QCD: $G = SU(3)$

$$\psi_A = \psi_A^i \quad ; \quad i = (\text{green}, \text{red}, \text{blue})$$

$$T_a = \frac{1}{2} \lambda_a$$

Gell-mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

$$\mathcal{L}_m = i\bar\Psi(\not{D} - im)\Psi \doteq i\bar\Psi^{(u)}(\not{D}^{(u)} - im)\Psi^{(u)}$$

$\gamma^\mu(\partial_\mu + ig A_\mu)$

gauge coupling

$$\Psi^{(u)}_{(x)} = U(x)\Psi(x)$$

$$\bar\Psi^{(u)}_{(x)} = \bar\Psi(x)\underbrace{U^+(x)}_{=U^{-1}(x)}$$

$$\Rightarrow \not{D}^{(u)} = U^{+-1}\not{D}\,U^{-1}$$

$$\Rightarrow igA^{(u)}_\mu = \underbrace{U^{+-1}_{(x)}\partial_\mu U(x)}_{\text{cf. QED}} + \underbrace{iU^{+-1}A_\mu U^{-1}}_{\text{New}}$$

$$\simeq i\partial_\mu\theta^a T_a + i[A_\mu, \theta^a T_a]$$

$$U(x) \simeq \mathbb{1} - i\theta^a_{(x)} T_a$$

cf. Problem sheet 2.