The Quantum BV-equation:

Rep: Anti-bracket:

$$(A,B) = \int \frac{\delta_R A}{\delta \Phi^I} \frac{\delta_L B}{\delta \Phi^{*I}} - \frac{\delta_R A}{\delta \Phi^{*I}} \frac{\delta_L B}{\delta \Phi^I}$$

In particular,

$$(S,S) = \int \frac{\delta_R S}{\delta \Phi^I} \frac{\delta_L S}{\delta \Phi^{*I}} - \frac{\delta_R S}{\delta \Phi^{*I}} \frac{\delta_L S}{\delta \Phi^I}$$

$$= \int \frac{\delta_R S}{\delta \Phi^I} \frac{\delta_L S}{\delta \Phi^{*I}} + \frac{\delta_L S}{\delta \Phi^{*I}} \frac{\delta_R S}{\delta \Phi^I}$$

$$= \int \frac{\delta_R S}{\delta \Phi^I} \frac{\delta_L S}{\delta \Phi^{*I}} + \frac{\delta_R S}{\delta \Phi^I} \frac{\delta_L S}{\delta \Phi^{*I}}$$

$= 0$ iff master equation is satisfied

A generic BV-action is then invariant under the generalised BRST tf:

$$\delta \Phi^I = \delta\xi (S, \Phi^I) \quad ; \quad \delta \Phi^{*I} = \delta\xi (S, \Phi^{*I})$$

$$= \delta\xi \underbrace{\frac{\delta_L S}{\delta \Phi^{*I}}}_{= -\frac{\delta_R S}{\delta \Phi^{*I}}} \qquad\qquad = \delta\xi \frac{\delta_R S}{\delta \Phi^I}$$

$(\cdot,\cdot)$ acts as a derivation, that is

$$(f, gh) = (f,g)\,h \pm g\,(f,h) \quad ; \quad f,g,h \in \mathcal{F}$$

Thus $\delta_\xi f = \delta\xi\,(S,f)$. In particular,

$$\delta_\xi S = (S,S) = 0$$

Furthermore,
$$\begin{aligned} s : \mathcal{F} &\longrightarrow \mathcal{F} \\ f &\longmapsto (S,f) \end{aligned}$$

is a nil-potent vector field on $\mathcal{F}$, since

$$(S,(S,f)) = -\tfrac{1}{2}\,(f,(S,S)) = 0$$

by the Jacobi identity: $\{f,(g,h)\} \pm$ cyclic perm. $= 0$

(Exercise)

This then implies that if $(S,S) = 0$ then

$S' = S + (S,f)$ also satisfies the master equation. In particular, for $f = \varepsilon\delta\psi(\Phi)$

small g.f. fermion

$$S'[\Phi, \Phi^*] = S[\Phi, \Phi^*] + \varepsilon \int \frac{\delta_L \delta\psi}{\delta \Phi^I} \frac{\delta_R S}{\delta \Phi^{*I}}$$

$$= S[\Phi, \Phi^*] - \varepsilon \int \frac{\delta_R S}{\delta \Phi^{*I}} \frac{\delta_L \delta\psi}{\delta \Phi^I}$$

$$= S[\Phi, \Phi^* - \varepsilon \frac{\delta \delta\psi}{\delta \Phi^I}] \quad , \quad (*)$$

Let us now consider the variation of the quantum partition function

$$Z_\psi = \int [D\Phi] \, e^{\frac{i}{\hbar} S[\Phi^I, \Phi^{*I} = \frac{\delta\psi}{\delta \Phi^I}]}$$

under a change of the gauge fixing fermion $\psi \to \psi + \delta\psi$. According to $(*)$

$$\delta Z = -\left(\frac{i}{\hbar} \int [D\Phi] \, e^{\frac{i}{\hbar} S[\Phi, \Phi^*]} \int \frac{\delta_R S}{\delta \Phi^{*I}} \frac{\delta_L(\delta\psi)}{\delta \Phi^I} \right)_{\Phi^* = \frac{\delta\psi}{\delta\phi}}$$

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$$= -\left(\frac{1}{\hbar} \int [D\Phi] \, e^{\frac{i}{\hbar} S[\Phi, \Phi^*]} \int \Big\{ \underbrace{\frac{1}{\hbar} \frac{\delta_R S}{\delta \Phi^{*I}} \frac{\delta_L S}{\delta \Phi^I}}_{= \frac{1}{2\hbar}(S, S)} - i \underbrace{\frac{\delta_L}{\delta \Phi^I} \frac{\delta_R}{\delta \Phi^{*I}} S}_{=: \Delta S} \Big\} \delta\psi(\Phi) \right)_{\Phi^* = \frac{\delta\psi}{\delta\phi}}$$

$$= -\frac{1}{2\hbar^2} \int [D\Phi]\, e^{\frac{i}{\hbar} S[\Phi,\Phi^*]} \underbrace{\left\{ (S,S) - 2i\hbar \Delta S \right\}}_{\boxed{\text{Quantum BV equation}}} \delta\Psi(\phi) \Big|_{\Phi^* = \frac{\delta\Psi}{\delta\Phi}}$$

In order to develop some intuition about this equation let us consider the concrete example where

$$\Psi = -\bar{H}_a \left(\frac{1}{2\lambda} b_a + F_a \right)$$

$$\delta\Psi = -\bar{H}_a\, \delta F_a$$

Then,

$$\int \frac{\delta_R S}{\delta \Phi^{*I}} \frac{\delta_L(\delta\Psi)}{\delta\Phi^I} = \int - \underbrace{\frac{\delta_R S}{\delta \bar{H}^*_a}}_{= b_a} \delta F_a - \underbrace{\frac{\delta_R S}{\delta A^*_\mu}}_{s(A_\mu)} \bar{H}_a \frac{\delta F_a}{\delta A_\mu}$$

Thus, the quantum BV equation implies that

$$\delta_\xi Z_\Psi = +\frac{i}{\hbar} \int [DA] \left(F_a \delta F^a + S\Delta [A] \right) e^{\frac{i}{\hbar} S^{tot}}$$

$$= 0$$

which implies, in particular, the S-T identity.