An alternative representation of $\tilde{I}$ which avoids using the equation of motion is given by:

$$\tilde{I}[p_\mu, q^\nu, b, c] = \int \left( p_\nu \dot{q}^\nu - \frac{1}{2}(p^\mu p_\mu + m^2) + i b \dot{c} \right) d\tau$$

with

$$\delta q^\mu = -i p^\mu c, \quad \delta b = H, \quad \delta c = \delta p_\mu = 0$$

Next we want to identify a representation space $H$, for the operator $\{p_\mu, q^\nu, b, c\}$:

We can represent these operators on a graded space of function with $\mu = 0,1,2,3$

$$p_\mu = \frac{1}{i}\partial_\mu \quad ; \quad b = \frac{\partial_L}{\partial c}$$

a generic state $\psi \in H$ has the expansion

$$\psi = \phi(q^\mu) + \phi^*(q^\mu)\, c$$

From this we are lead to identify $f_1(q^\mu)$ with of a scalar field. To justify this we recall

$$\langle \phi_2 | e^{-i(\tau_f - \tau_i)\mathcal{H}} | \phi_1 \rangle =$$

$$\int dq^\mu_{f2} \int dq^\nu_{f2} \langle q^\mu_{f2} | e^{-i(\tau_f - \tau_i)\mathcal{H}} | q^\mu_{f1} \rangle$$

$$= \int dq^\mu_{f2} \int dq^\nu_{f1} \, K(q^\mu_{f2}, q^\nu_{f1}, \tau_f, \tau_i)$$

so that $\phi(q^\mu)$ describes a superposition of "initial" (at $\tau = \tau_i$) and "final" ($\tau = \tau_f$) "states" of the scalar particle

This makes also clear that the "state" $\phi_1$ is preserved under the evolution generated by $\mathcal{H}$ iff $\mathcal{H}\phi_1 = 0 \Leftrightarrow (p^2 + m^2)\phi_1 = 0$ which are just the (Klein Gordon) equations of motion for a free scalar field.

On the other hand $\mathcal{H}\phi = 0 \Rightarrow Q\phi = 0$. Furthermore there is no state in $\mathcal{H}$ with degree $-1$. Thus $\phi \in coh(Q)|_{deg=0}$ i.e. an on shell massive scalar field is in $coh(Q)|_{deg\,0}$.

$\phi^*$ on the other hand is identified with the BV - antifield for $\phi$.

Next we want to construct the BV action for. For this we need to define an inner product on $H$. Such a pairing is obtained as

$$\langle \psi_1 | \psi_2 \rangle = \lim_{T \to 0} K(T, \psi_1, \psi_2)$$

where

$$K(T, \psi_1, \psi_2) \equiv \int d^4q_1 \, d^4q_2 \, \langle q_1 | \bar{\psi}_1(q) \, e^{-iTH} \psi_2(q_2) | q_2 \rangle$$

$$= \int d^4q_1 d^4q_2 \int_{q_2}^{q_1} [D q, b, c] \, \psi_1(q_1(T)) \, \psi_2(q_2(0)) \, e^{\frac{i}{\hbar} \tilde{I}[q, b, c]}$$

For $T \to 0$ the path integral reduces to

$\int dc \, \delta^4(q_1 - q_2)$ where $\int dc$ is the integral over the constant ghost mode which is not supressed in the action. Thus

$$\langle \psi_1 | \psi_2 \rangle = \int d^4q \int dc \, \bar{\psi}_1(q, c) \, \psi_2(q, c)$$

Rep: $\psi_{1/2}(q) = \phi_{1/2}(q) + \phi^*_{1/2}(q) \, c$

Thus $\langle , \rangle$ is an <u>odd</u> pairing between $\Psi|_{deg=0}$ and $\Psi|_{deg=1}$, or and <u>odd symplectic form</u>. With the help of $\langle , \rangle$ we can then define an action for $\Psi$ as

$$S[\Psi] = \frac{1}{2}\langle \Psi | Q | \Psi \rangle = \frac{1}{2}\int (\bar{\phi} + \bar{\phi}^* c)\, c\, \mathcal{H} (\phi + \phi^* c)$$

$$= \frac{1}{2}\int \bar{\phi}\, \mathcal{H}\, \phi = \frac{1}{2}\int \left( \partial\phi\, \partial\bar{\phi} + m^2 \phi \bar{\phi} \right)$$

which is just the Klein-Gordon action, which upon variation w.r.t. $\Psi$, reproduces the equations of motion for $\phi(q)$ which is in this model the physical state condition for $Q$.

<u>In</u> sum: The Klein-Gordon action is the BV-action for the reparametrisation invariant world line.