More generally, we can assume that $S'$ is a non-linear functional of the antifields $\phi^{*I}$ with an expansion of the form

$$S'[\phi, \phi^*] = S_0[A_\mu] + \int \underset{(1)}{H^a} \underset{(0)}{f_a^r[A]} \underset{(-1)}{A^{*r}}$$

$$+ \frac{1}{2}\int \underset{(1)}{H^a}\underset{(1)}{H^b} \underset{(0)}{f^c_{ab}[A]} \underset{(-2)}{H^{*c}}$$

$$+ \frac{1}{2}\int H^a H^b f^{rs}_{ab}[A] A^{*r} A^{*s}$$

$$+ \dots$$

Then the master equation $0 \overset{!}{=} \int \frac{\delta_R S'}{\delta \phi^{*I}} \frac{\delta_L S'}{\delta \phi^I}$ gives

$O(\phi^{*0})$: $\int \underbrace{H^a f^I_a[A]}_{\delta A^I = s(A^I)} \frac{\delta S[A]}{\delta A^I} = 0$ $\Rightarrow$ gauge invariance of $S_0[A_\mu]$ with $\theta^a \rightsquigarrow H^a$

$$O(\phi^*): \left.\begin{aligned} &\int H^a f^r_a(A) H^b \frac{\delta f^s_b(A)}{\delta A^r} A^{*s} \\ &+ \int \frac{1}{2} H^a H^b f^c_{ab}[A] f^s_c(A) A^{*s} \end{aligned}\right\} \Leftrightarrow [\delta_\alpha, \delta_\beta] = \delta_{[\alpha,\beta]}$$

$$+ \int H^a H^b f^{rs}_{ab}[A] A^{*r} \frac{\delta S}{\delta A^s} \Big\} = 0 \quad \text{for } f^{rs}_{ab} = 0 \text{ or } \frac{\delta S_0}{\delta A^s} = 0 \text{ e.m.}$$

Interpretation: if $f^{rs}_{ab} \neq 0$ then we have an open symmetry algebra (closes only mod. equations of motion)

This occurs, for example, in supergravity where the SUSY algebra closes only up to equations of motion.

For open algebras $s^2 \neq 0$ but BV can still be applied.

The BV-generalisation of the BRST formalism is also required when the gauge invariance is reducible, i.e

$$A_\mu \sim A_\mu + s(A_\mu) = A_\mu + D_\mu H$$

with $H \sim H + s(H)$

which requires to introduce ghosts for the ghosts (e.g. string field theory or gerbes (p-form gauge fields) (see below)

For a generic B-V action the master equation expresses the invariance of $S'$ under a generalised BRST transformation. In order to see this we first define the antibraket. If F and G are differentiable functions on $\mathcal{F}$ then

$$\boxed{(F, G) \equiv \int \frac{\delta_R F}{\delta \phi^I} \frac{\delta_L G}{\delta \phi^{*I}} - \frac{\delta_R F}{\delta \phi^{*I}} \frac{\delta_L G}{\delta \phi^I}}$$

Then $\delta \Phi^I = (S, \phi^I)$ and $\delta \Phi^{*I} = (S, \phi^{*I})$ is the generalised BRST symm. and $\delta S = (S, S) = 0$ by the BV-master equation (Exercise)

Illustration: 2-form gauge field, $A^{[2]}_{\mu\nu} = -A^{[2]}_{\nu\mu}$

Fieldstrength: $F^{[3]}_{\mu\nu\chi} = \partial_{[\mu} A^{[2]}_{\nu\chi]}$, or $F^{[3]} = dA^{[2]}$

Action: $S_0[A^{[2]}] = -\frac{1}{2}\int F^{[3]\mu\nu\chi} F^{[3]}_{\mu\nu\chi} = -\frac{1}{2}\int F^{[3]} \wedge \underset{\text{hodge dual}}{*} F^{[3]}$

gauge invariance: $A^{[2]}_{\mu\nu} \mapsto A^{[2]}_{\mu\nu} + \partial_{[\mu}\sigma^{[1]}_{\nu]} \sim A_{\mu\nu}$

or $\quad A^{[2]} \mapsto A^{[2]} + d\underbrace{\sigma^{[1]}}_{\text{1-form}}$

Furthermore $\sigma^{[1]}_{\nu} \sim \sigma^{[1]}_{\nu} + \partial_\nu \sigma^{[0]}$

$\sigma^{[1]} \sim \sigma^{[1]} + d\sigma^{[0]}$

gauge for gauge: reducible gauge inv.

In the BV formulation we introduce $\underset{(1)}{H^{[1]}}$ for $\sigma^{[1]}$ and $\underset{(2)}{H^{[0]}}$ for $\sigma^{[0]}$ $\Big\}$ $\{\Phi^I\} = \{\underset{(0)}{A^{[2]}}, \underset{(1)}{H^{[1]}}, \underset{(2)}{H^{[0]}}\}$

Antifields: $\{\Phi^{*I}\} = \{\underset{(-1)}{A^{[2]*}_{\mu\nu}}, \underset{(-2)}{H^{[1]*}}, \underset{(-3)}{H^{[0]*}}\}$

BV-action: $S = S_0 + \underbrace{\int A^{[2]*\mu\nu}\partial_\mu H^{[1]}_\nu}_{= \int A^{[2]*}\wedge * dH^{[1]}} + \underbrace{\int H^{[1]*\mu}\partial_\mu H^{[0]}}_{H^{[1]*}\wedge * dH^{[0]}}$

In order to fix the gauge we now introduce 2 <u>trivial pairs</u>

$$\big(\underset{(-1)}{\bar{H}^{[1]}}, \underset{(0)}{b^{[1]}}\big) \quad \text{and} \quad \big(\underset{(-2)}{\bar{H}^{[0]}}, \underset{(-1)}{b^{[0]}}\big)$$

together with their <u>anti-fields</u>:

$$\big(\underset{(0)}{\bar{H}^{[1]*}}, \underset{(-1)}{b^{[1]*}}\big) \quad \text{and} \quad \big(\underset{(1)}{\bar{H}^{[0]*}}, \underset{(0)}{b^{[0]*}}\big)$$

and invariant action

$$S_t = \int \bar{H}^{[1]*\mu} \, b^{[1]}_{\mu} + \int \bar{H}^{[0]*} \, b^{[0]}$$

Finally we introduce a gauge-fixing fermion to eliminate the anti-fields.

$\boxed{\text{Exercise}}$