Rep: Gauß law and Gupta-Bleuler formulation
(Operator formalism)

① $\int \theta(y) \{ \underline{\nabla} \cdot \underline{E}(y), \underline{A}(x) \} = \underline{\nabla} \theta(x)$

② $(\underline{\nabla} \cdot \underline{E})^{-} |phys\rangle = \langle phys| (\underline{\nabla} \cdot \underline{E})^{+} = 0 \Rightarrow$
no longitudinal state

③ $\underline{\nabla} \cdot \underline{E} |0\rangle$ is a longitudinal state

Equivalence classes: $|phys\rangle \sim |phys\rangle + \int \theta (\nabla E) |0\rangle$

in the sense that for $[G, \underline{\nabla} \cdot \underline{E}] = 0 \Leftrightarrow G$ observable

$$\langle phys| G \left(|phys\rangle + \int \theta \, \underline{\nabla} \cdot \underline{E} |0\rangle \right)$$

$$= \langle phys| G |phys\rangle$$

BRST in operator formalism

$$\underline{\nabla} \underline{E} \rightsquigarrow Q \simeq \int H^a (\underline{\nabla} \cdot \underline{E})^a + b^a \Pi_{\bar{H}^a}$$
$$+ \frac{1}{2} \Pi_{H^a} C_{abc} H^b H^c$$

$[Q, \phi] = i \mathfrak{s}(\phi) \rightarrow$ BRST vector field

$$
Q\,|\text{phys}\rangle = \Big(\overset{\overset{0}{\text{no ghost}}}{\cancel{H}^-}(\underline{\nabla}\cdot\underline{E})^+ + H^+ \overset{\overset{0}{\text{no longid. ph.}}}{\cancel{(\underline{\nabla}\cdot\underline{E})^-}} + \overset{\overset{0}{\text{no longid pho / no timelike ph}}}{\cancel{b}^+}(\partial_0 H)^- + b^-(\partial_0 \overset{\overset{0}{\text{no ghost}}}{\cancel{H}})^+ \Big)|\text{phys}\rangle
$$

$$
+ \Big(\tfrac{1}{2}\overset{\overset{0}{\text{no antighost}}}{\cancel{(\partial_0 \bar{H}^a)^+}}(C_{abc}H^bH^c)^- + \tfrac{1}{2}(\partial_0 \bar{H})^-(C_{abc}\overset{\overset{0}{\text{no ghost}}}{\cancel{H}^b}H^c)^+ \Big)|\text{phys}\rangle \overset{!}{=} 0
$$

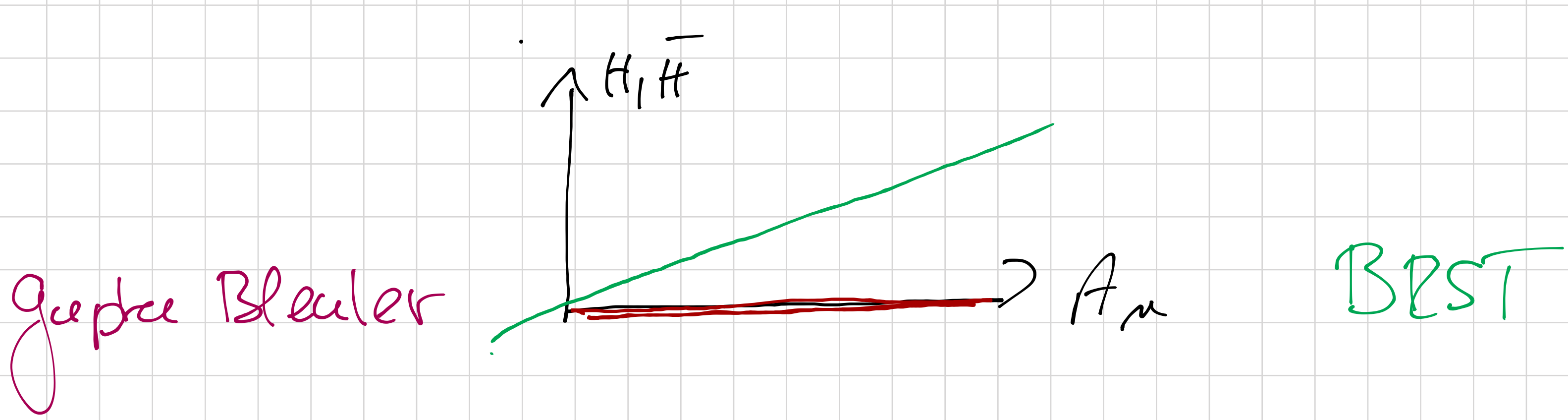
$$
b_a = F_a = (\partial A)^a = \partial_0 A^0 - \underline{\nabla}\cdot\underline{A}
$$

Thus $Q\,|\text{phys}\rangle = 0 \Longleftarrow$

- $(\underline{\nabla}\cdot\underline{E})^-|\text{phys}\rangle = 0 \Rightarrow$ no longitudinal state
- $H^-|\text{phys}\rangle = 0 \Rightarrow |\text{phys}\rangle \not\ni \{$ghosts or anti-ghosts$\}$
- $\partial_0 A^0|\text{phys}\rangle = 0 \Rightarrow$ no timelike photon.

Rem ① In the BRST formulation there is no need to distinguish $(\underline{\nabla}\cdot\underline{E})^+$ and $(\underline{\nabla}\cdot\underline{E})^-$ since they combine with the ghosts s.t. $Q|\text{phys}\rangle = 0$

② choice of representative:



Put differently: Since in the BRST formulation there are more degrees of freedom there is more freedom to embed the physical states.

③ While $(\underline{\nabla}\cdot\underline{E})^2 \neq 0$ we have $Q^2 \equiv 0$.

relation to differential forms: $df = \partial_i f\, dx^i$

$$d^2 f = \partial_i \partial_j f \underbrace{dx^i \wedge dx^j}_{\text{anticommuting.}} \equiv 0$$

so, Q behaves like an external differential.

[The precise geometric interpretation of Q (or s) is that of an <u>odd</u> vector field on $\mathcal{F}$

Rep: $v = v^i \partial_{x^i} \in TM$ vector field (manifold)

$$\underset{\text{odd}}{s} = \underset{\substack{\text{odd}\\ \text{even}}}{s^i} \underset{\substack{\text{even}\\ \text{odd}}}{\partial_{x^i}} \in \Pi TM$$

(parity) twisted tangent bdl.]

Irrespective, since $Q^2 \equiv 0$ we may decompose the space of states $\mathcal{H}$ into Q-closed and non-Q-closed states, with equivalence classes: Let $|\psi\rangle$ s.t. that $Q|\psi\rangle = 0$ then $|\psi\rangle \sim |\psi\rangle + Q|\theta\rangle$

Def: The cohomology of Q is defined as

$$\mathrm{coh}(Q) = \{a \in \mathcal{H} \mid Qa = 0\} / \{a = Q\lambda\}$$

It turns out that $\mathrm{coh}(Q)$ are precisely the physical (i.e. gauge-invariant) states in $\mathcal{H}$.

Indeed for $|\alpha\rangle \in \mathcal{H}_{in}$ and $|\beta\rangle \in \mathcal{H}_{out}$ we have

$$\delta_\psi \langle\beta|\alpha\rangle = \frac{i}{\hbar}\langle\beta| s(\delta\psi)|\alpha\rangle = \frac{1}{\hbar}\langle\beta|[Q, \delta\psi]|\alpha\rangle$$

since $s(\delta\psi)$ is not zero, in general we find that the transition amplitude $A_{\beta\alpha} = \langle\beta|\alpha\rangle$ is independent of the choice of gauge, ψ iff $Q|\alpha\rangle = Q|\beta\rangle = 0$.

On the other hand $|\alpha\rangle$ and $|\alpha\rangle + Q|\lambda\rangle$ have the same amplitudes for all $|\beta\rangle$ with $Q|\beta\rangle = 0$, and therefore $|\alpha\rangle$ and $|\alpha\rangle + Q|\lambda\rangle$ are indistinguishable. We therefore identify

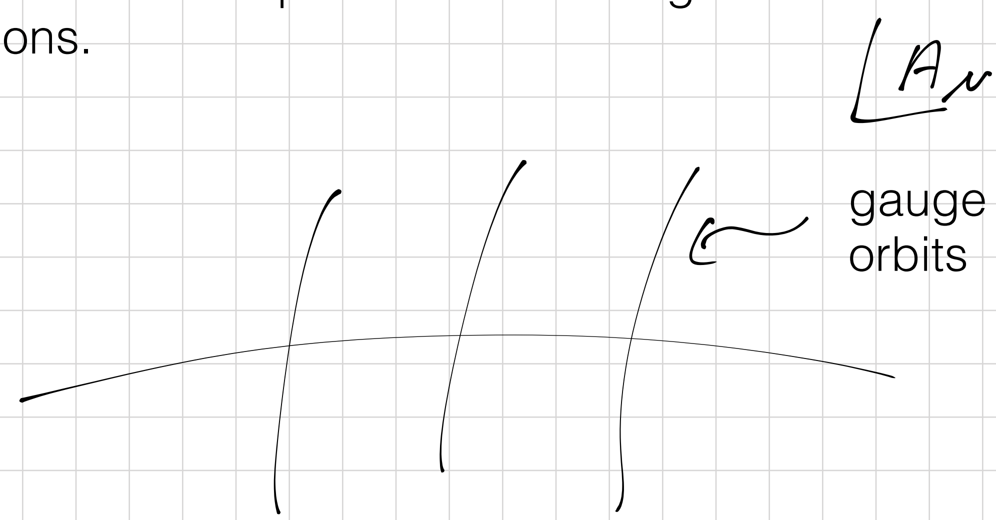
$$|\alpha\rangle \sim |\alpha\rangle + Q|\lambda\rangle \quad \forall \lambda$$

In other words, the physical states (i.e. the states that give gauge-inv. transition amplitudes) are in $\mathrm{coh}(Q, \mathcal{H})$. Similarly, correlation functions $\langle f(A_\mu)\rangle$ are gauge invariant if $[Q, f(A_\mu)] = 0$.

In closing our discussion of the BRST quantisation, let us return to the geometric interpretation of what we have done. Starting with the ill-defined path integral

$$\int D[A]\; f(A_\mu)\; e^{\frac{i}{\hbar} S[A_\mu]}$$

which formally integrates over cell vector potentials including orbits over gauge equivalent configurations.



we factorised the pure gauge contributions $\quad \int D[g]$

with the help of the Faddeev-Popov trick and then replaced

$$\int D[g] \quad \text{by} \quad \int D[\eta,\eta^*]$$

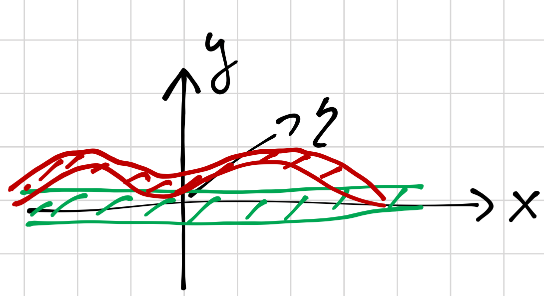
which has the advantage of

$$\int D[A,\eta^*,\eta]\;\; \delta[F^a]\; e^{\frac{i}{\hbar} S(A_\mu) + \eta^* M \eta}$$

being finite and furthermore independent on the choice of gauge fixing

Illustration: $\quad \int dx\, e^{-x^2} = \int dx\,dy\;\; \delta(f(y))\,|f'(0)|\, e^{-x^2}$

$$= \int dx\,dy\,d\eta^* d\eta\; \delta(f(y))\, e^{-x^2+\eta^*|f'(0)|\eta}$$



more generally, $\underline{f(y)} \to f(y,x)$

n. b: only tangent space at zero of f is relevant