Let us elaborate on this:

Rep: For the e.m. field we have

$$\underline{\Pi}_{\underline{A}} = \underline{E} \quad \text{with} \quad \{ A_i(\underline{x},t)\, E_j(\underline{y},t) \} = \delta^3(\underline{x}-\underline{y})$$

Then

$$\delta A_i(\underline{x},t) = \int d^3y\, \Theta(\underline{y},t) \{ A_i(\underline{x},t), \underbrace{\underline{\nabla}\cdot\underline{E}(\underline{y},t)} \}$$

Gauß constraint

$$= \int d^3y\, \Theta(\underline{y},t)\, \nabla_{y_i} \delta^3(\underline{y}-\underline{x})$$

no bdy $\rightarrow$ $= -\partial_{y_i} \Theta(\underline{y},t)$ ✓

$\rightarrow$ replace $\Theta(\underline{y},t)$ by $H^a(\underline{y},t)$

$\underline{E}(\underline{y},t)$ $\quad\quad$ $\underline{E}^a(\underline{y},t)$

ghosts:

$$\Pi_{H^a}(\underline{x},t) = h \frac{\overleftarrow{\delta}}{\delta \partial_0 H(\underline{x},t)} = -\partial_0 \bar{H}_a(\underline{x},t)$$

$$\Pi_{\bar{H}_a}(\underline{x},t) = \frac{\overrightarrow{\delta}}{\delta \partial_0 \bar{H}_a(\underline{x},t)} h = \partial_0 H^a(\underline{x},t)$$

with

$$\{ H^a(\underline{x},t), \Pi_{H^b}(\underline{y},t) \}_+ \equiv \delta^a_b\, \delta^3(\underline{x}-\underline{y})$$

symmetric

$$\{ \bar{H}_a \quad \Pi_{\bar{H}^b} \}_+ = \delta_a{}^b\, \delta^3(\underline{x}-\underline{y})$$

Then $Q = \int d^3y \, \{ H^a(\underline{y},t) \, \underline{\nabla} \cdot \underline{E}^a(\underline{y},t)$

$+ b^a(\underline{y},t) \, \Pi_{\bar{H}^a}(\underline{y},t)$

$+ \Pi_{H^a}(\underline{y},t) \, C_{abc} H^b(\underline{x},t) H^c(\underline{x},t) \}$

reproduces (++) throug $s(\Phi) = \{Q, \Phi\}_\pm$

In the quantum theory $\{ , \}$ is replaced by the commutator and $\{ , \}_+$ by the anti-commutator.

Furthermore, $Q^2 = [Q, Q] = 0$.

Thus Q is a nilpotent operator on the Hilbert space of states

Rem: Q acts on quantum fields with a (anti) commutator

$Q : \Phi \longmapsto [Q, \Phi]_\pm$

$\longrightarrow$ adjoint action

At the same time Q acts on the Hilbertspace $\mathcal{H}$ as a <u>linear</u> operator

$$Q: \mathcal{H} \longrightarrow \mathcal{H}$$
$$a \longmapsto Qa$$

<u>Example:</u> ED: $\underline{E}(\underline{x},t)$ configuration (trajectory in mechanics)

QFT:

transverse + longitud. + time-like

$$|\underline{k}, e\rangle = \sum_{\lambda=1}^{4} \underline{e}(\underline{k}\,\lambda)\, a(\underline{k},\lambda)$$

= photon with wave vector $\underline{k}$ and polarisation $\underline{e}(\underline{k}) = \sum_{\lambda} \underline{e}(\underline{k}\,\lambda)$

$$\underline{\nabla}\cdot\underline{E} \sim \int d^3p \sum_{\lambda=1}^{4} \underline{e}(\underline{p},\lambda)\cdot\underline{p}\,\Big(a(p,\lambda)\, e^{-ik\cdot x} + a^+(p,\lambda) e^{ikx}\Big)$$

$$\underline{\nabla}\cdot\underline{E}\ |\underline{k}, e(k)\rangle =$$

Gupta-Bleuler

$$\int d^3p \sum_{\lambda} \underline{e}(p,\lambda)\cdot\underline{p}\,\Big(a(\underline{p},\lambda) e^{-ikx} + a^+(p,\lambda) e^{ikx}\Big) \sum_{\lambda'} \underline{e}(\underline{k},\lambda')\, a^+(k,\lambda')\,|0\rangle$$

$$\delta^3(\underline{p}-\underline{k})\,\delta_{\lambda\lambda'}$$

$$= \sum_{\lambda} \underline{e}(k,\lambda)\cdot\underline{k}\; e(k,\lambda)\, e^{-ik\cdot x}$$

$$= \underline{k}\cdot\left(\underbrace{\sum_{\lambda=1}^{2} e(\underline{k},\lambda)\, e(\underline{k},\lambda)}_{= g_{\mu\nu} - \frac{k_\mu k_\nu}{|\underline{k}|^2} - n^0\otimes n^0} + \underbrace{\sum_{\lambda=3}^{4} e(k,\lambda) e(k,\lambda)}_{= \frac{k_\mu k_\nu}{|\underline{k}|^2},\; n^0\otimes n^0}\right)$$

transverse

time-like

longitudinal

$= k_\nu - k_\nu = 0$

$= k_\nu \neq 0$

Thus: $\underline{\nabla}\cdot\hat{\underline{E}}\;|\underline{k},\underline{e}\rangle = 0 \Rightarrow$ no longitudinal photons

[But: $\underline{\nabla}\cdot\underline{E}\,|0\rangle$ is a longitudinal state]

(due to $\Theta \to \mathcal{H}$)

Now, recalling that $Q^2 \equiv 0$ on $\mathcal{H}$, in analogy with the de Rham differential d, we can analyse