# Problem Set 5:

Handout: Fri, Jun 19, 2020; Solutions: Fri, Jul 03, 2020

**Problem 1** (counts as 3 problems) Kondo model

Consider the Kondo Hamiltonian,

$$\hat{\mathcal{H}}_{\mathrm{K}} = \int d^3\boldsymbol{k} \sum_{\sigma} \epsilon(k)\hat{c}^\dagger_{\boldsymbol{k},\sigma}\hat{c}_{\boldsymbol{k},\sigma} + J\hat{\boldsymbol{S}}\cdot\hat{\boldsymbol{S}}_e(\boldsymbol{0}), \tag{1}$$

with the band electron spin at the location $\boldsymbol{r}=\boldsymbol{0}$ of the Kondo spin $\hat{\boldsymbol{S}}$:

$$\hat{\boldsymbol{S}}_e(\boldsymbol{0}) = (2\pi)^{-3}\int d^3\boldsymbol{k}d^3\boldsymbol{k}' \sum_{\alpha,\beta}\hat{c}^\dagger_{\boldsymbol{k},\alpha}\frac{1}{2}\boldsymbol{\sigma}_{\alpha,\beta}\hat{c}_{\boldsymbol{k}',\beta} \tag{2}$$

(1.a) By an expansion into plane waves, show that the Kondo problem reduces to a 1D Hamiltonian $\hat{\mathcal{H}}^{1D}_{\mathrm{K}}$ which decouples from the rest of the system:

$$\hat{\mathcal{H}}_{\mathrm{K}} = \hat{\mathcal{H}}^{1D}_{\mathrm{K}} + \hat{\mathcal{H}}'_{\mathrm{K}}. \tag{3}$$

Derive an expression for $\hat{\mathcal{H}}'_{\mathrm{K}}$ and show that

$$\hat{\mathcal{H}}^{1D}_{\mathrm{K}} = \int_0^{\Lambda_{\mathrm{UV}}} dk \sum_{\sigma}\epsilon(k)\ \hat{s}^\dagger_{k,\sigma}\hat{s}_{k,\sigma} + J\hat{\boldsymbol{S}}\cdot\hat{\boldsymbol{S}}_e(\boldsymbol{0}), \tag{4}$$

where:

$$\hat{\boldsymbol{S}}_e(\boldsymbol{0}) = \sum_{\alpha,\beta}\hat{s}^\dagger_\alpha(0)\frac{1}{2}\boldsymbol{\sigma}_{\alpha,\beta}\hat{s}_\beta(0), \qquad \hat{s}_\sigma(0) = \frac{1}{\sqrt{2}\pi}\int_0^{\Lambda_{\mathrm{UV}}} dk\ \hat{s}_{k,\sigma}. \tag{5}$$

(1.b) Linearizing the band Hamiltonian Eq. (4) around the Fermi energy, $\epsilon(k)\simeq\hbar k v_{\mathrm{F}}$, yields:

$$\hat{\mathcal{H}}^{1D}_{\mathrm{K}} \simeq \int_{-\infty}^{\infty} dk \sum_{\sigma}\hbar v_{\mathrm{F}} k\ \hat{s}^\dagger_{k,\sigma}\hat{s}_{k,\sigma} + J\hat{\boldsymbol{S}}\cdot\hat{\boldsymbol{S}}_e(\boldsymbol{0}). \tag{6}$$

This Hamiltonian can be *bosonized* by defining spin- and charge- density operators,

$$\hat{\rho}(k) = \sum_{\sigma=\pm}\int_{-\infty}^{\infty} dp\ \hat{s}^\dagger_{p+k,\sigma}\hat{s}_{p,\sigma}, \qquad \hat{\rho}(-k) = \hat{\rho}^\dagger(k) \tag{7}$$

$$\hat{\sigma}(k) = \sum_{\sigma=\pm}\int_{-\infty}^{\infty} dp\ \sigma\ \hat{s}^\dagger_{p+k,\sigma}\hat{s}_{p,\sigma}, \qquad \hat{\sigma}(-k) = \hat{\sigma}^\dagger(k). \tag{8}$$

Calculate the *commutation relations* of $\hat{\rho}$ and $\hat{\sigma}$, assuming a band with all states at $p<0$ occupied. Show that they define *bosonic operators*, $\hat{\rho}\propto\hat{b}_{-k}$ and $\hat{\sigma}(k)\propto\hat{a}_{-k}$. For the new bosonic operators we will show in the tutorial that:

$$\hat{\mathcal{H}}^{1D}_{\mathrm{K}} = \hbar v_{\mathrm{F}}\int_0^{\infty} dk\ k\ \left(\hat{a}^\dagger_k\hat{a}_k + \hat{b}^\dagger_k\hat{b}_k\right) + J\hat{\boldsymbol{S}}\cdot\hat{\boldsymbol{S}}_e(0). \tag{9}$$

(1.c) The Kondo interaction can be written as:

$$J\hat{\boldsymbol{S}}\cdot\hat{\boldsymbol{S}}_e(0) = \frac{J_z}{2}\hat{S}^z\sum_{\sigma=\pm}\sigma\ \hat{s}^\dagger_\sigma(0)\hat{s}_\sigma(0) + J_\perp\left[\hat{S}^+\hat{s}^\dagger_-(0)\hat{s}_+(0) + \text{h.c.}\right], \tag{10}$$

with $J_z = J_\perp = J$. Express the $J_z$-term by the new bosonic operators $\hat{a}_k$ and $\hat{b}_k$. As usual, $\hat{S}^\pm = \hat{S}^x \pm i\hat{S}^y$.

(1.d) Show that the operators

$$\hat{\psi}_\sigma(x) = (2\pi a)^{-1/2}\exp\left[\hat{j}_\sigma(x)\right], \qquad \text{with} \tag{11}$$

$$\hat{j}_\sigma(x) = \int_0^\infty dk\ e^{-ak/2}\ C_k\left(\hat{b}_k + \sigma\hat{a}_k\right)e^{i\sigma kx} - \text{h.c.} \tag{12}$$

and an appropriately defined normalization constant $C_k = \alpha/\sqrt{k}$ (to be determined), obey *fermionic anti-commutation relations* for given spin $\sigma$:

$$\{\hat{\psi}_\sigma(x), \hat{\psi}^\dagger_\sigma(x')\} = \delta(x - x'). \tag{13}$$

In the above expression, $a$ defines a short-distance cut-off which may be sent to $a \to 0$ in the end.

*Hint:* Show first that $[\hat{j}_\sigma(x), \hat{j}_{\sigma'}(y)] = -i\pi\sigma\ \mathrm{sgn}(x - y)\ \delta_{\sigma,\sigma'}$.

*Note:* To obtain full fermionic anti-commutations, also between different spins $\sigma \neq \sigma'$, one needs to include additional zero-modes in the representation (11). For simplicity we discard them now.

(1.e) You may now identify the fermionic operators $\hat{s}_\sigma(0) \equiv \hat{\psi}_\sigma(0)$. Using this relation, express the $J_\perp$-part of the Kondo interaction in Eq. (10) by the bosonic fields $\hat{a}_k$ and $\hat{b}_k$. Show that the interaction decouples from the $\hat{b}_k$ operators – i.e. only the spin channel described by $\hat{a}_k$ couples to the Kondo impurity.

*Hint:* The result is:

$$J_\perp\hat{S}^+\hat{s}^\dagger_-(0)\hat{s}_+(0) = \frac{J_\perp}{2\pi a}\hat{S}^+e^{\hat{\xi}}, \qquad \hat{\xi} = \int_0^\infty dk\ e^{-ak/2}2C_k\left(\hat{a}_k - \hat{a}^\dagger_k\right). \tag{14}$$

(1.f) Show that the resulting Kondo Hamiltonian $\hat{\mathcal{H}}^a_{\mathrm{K}}$ for the interacting modes $\hat{a}_k$ is equivalent to a spin-boson model, by applying the unitary transformation: $\hat{U} = \exp[\hat{S}^z\hat{\xi}]$, i.e. show that:

$$\hat{U}^\dagger\hat{\mathcal{H}}^a_{\mathrm{K}}\hat{U} = \text{ spin-boson model}. \tag{15}$$

Derive the resulting spin-boson Hamiltonian explicitly.

**Problem 2** Effective polaron mass:

Consider the 3D Fröhlich Hamiltonian:

$$\hat{\mathcal{H}}_{\mathrm{F}} = \frac{\hat{\boldsymbol{p}}^2}{2M} + \int d^3\boldsymbol{k}\ \omega_{\boldsymbol{k}} \hat{a}^\dagger_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}} + \int d^3\boldsymbol{k}\ V_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\hat{\boldsymbol{x}}} \left( \hat{a}_{\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}} \right), \tag{16}$$

with $V_{\boldsymbol{k}} \propto \alpha^{1/2}$.

(2.a) Use Rayleigh-Schrödinger perturbation theory to first order in $\alpha$ and start from the unperturbed eigenstates $|\boldsymbol{q}\rangle|n_{\boldsymbol{k}} = 0\rangle$ to derive their renormalized energy:

$$E_0(\boldsymbol{q}) = \frac{\boldsymbol{q}^2}{2M} - \int d^3\boldsymbol{k}\ \frac{V_{\boldsymbol{k}}^2}{\omega_{\boldsymbol{k}} + \frac{\boldsymbol{k}^2}{2M} - \frac{\boldsymbol{k}\cdot\boldsymbol{q}}{M}}. \tag{17}$$

(2.b) Use Eq. (17) to derive a formal expression for the effective polaron mass $M^*$, to first order in $\alpha$. Assume that $V_{\boldsymbol{k}} = V_k$ and $\omega_{\boldsymbol{k}} = \omega_k$ are rotationally invariant and simplify the resulting integrals as far as possible.