FROM UNIVERSE

TO PLANETS

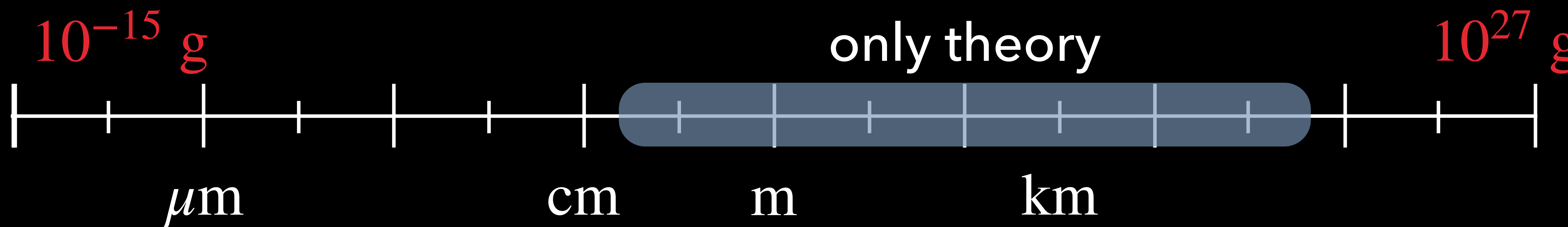
LECTURE 3

REVIEW: DUST SIZES AND MASSES

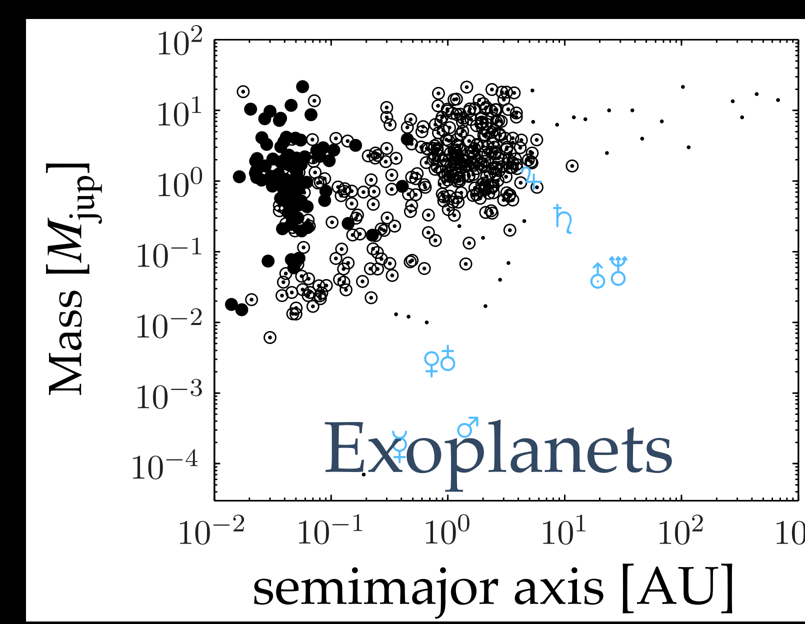
Samples

Lab & IDPs (interplanetary dust particles)

Meteorites



Observations: IR and (sub-)mm



OVERVIEW

Samples

Condensation

Lab & IDPs (interplanetary dust particles)

Meteorites

10^{-15} g

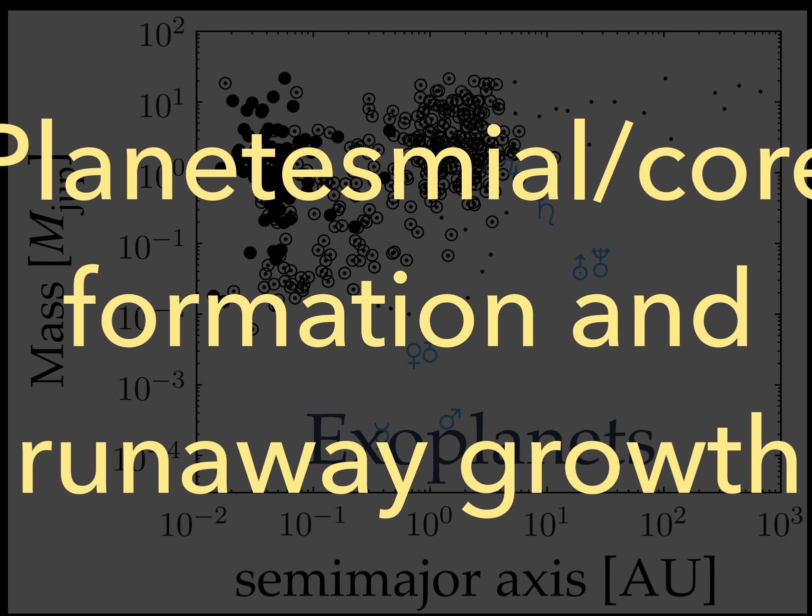
Collisional growth and fragmentation

only theory

10^{27} g

μm cm m km

Observations: IR and (sub-)mm



Planetesmial/core formation and runaway growth

FROM UNIVERSE

TO PLANETS

LECTURE 3.1: CONDENSATION

CONDENSATION

- Carbonaceous chondrites (a class of meteorites) show little chemical differentiation and fractionation (in contrast to, e.g. the Earth and Moon) ⟶ primitive. They provide clues to the initial chemical composition of the solar nebula.
    - Contain volatile organic chemicals and water, indicating that they have not undergone significant heating (>200 °C) since formation.
- CI-chondrites (the I is for Ivuna) are the most primitive sub-class. Contain H_2O (17–22%; bound in silicates), Fe (25%; in form of iron oxides), C (3–5%), amino acids, and PAHs.
    - Have not been heated above 50 °C (formed and remained beyond ~ 4 au).
    - Relative elemental abundances are similar to the Sun's photosphere. Notable exceptions are Li (used in nucleosynthesis) and volatile elements like H and O.

Stony-Iron Meteorite

Carbonaceous Chondrite

CI-Chondrite

CONDENSATION

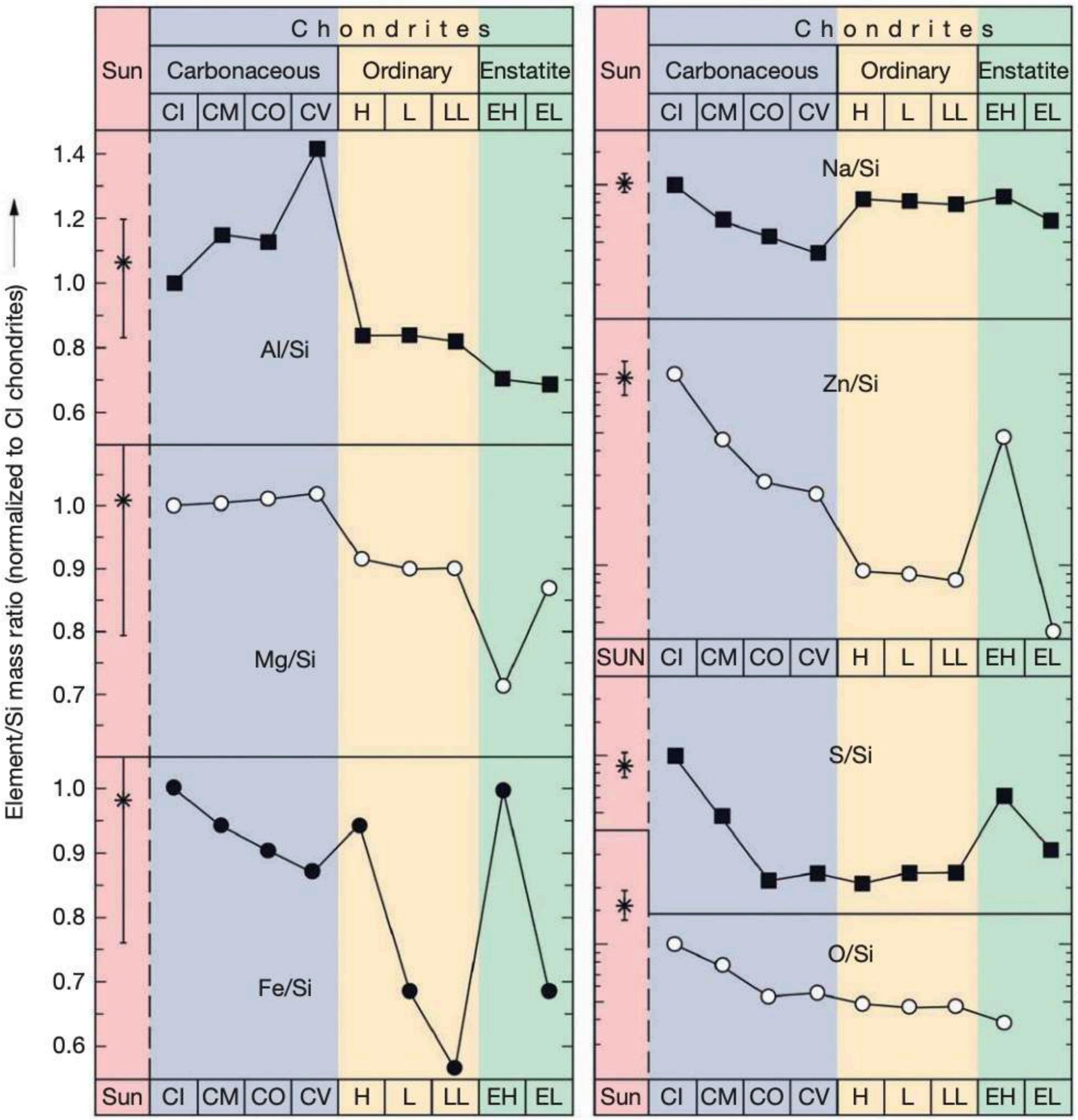
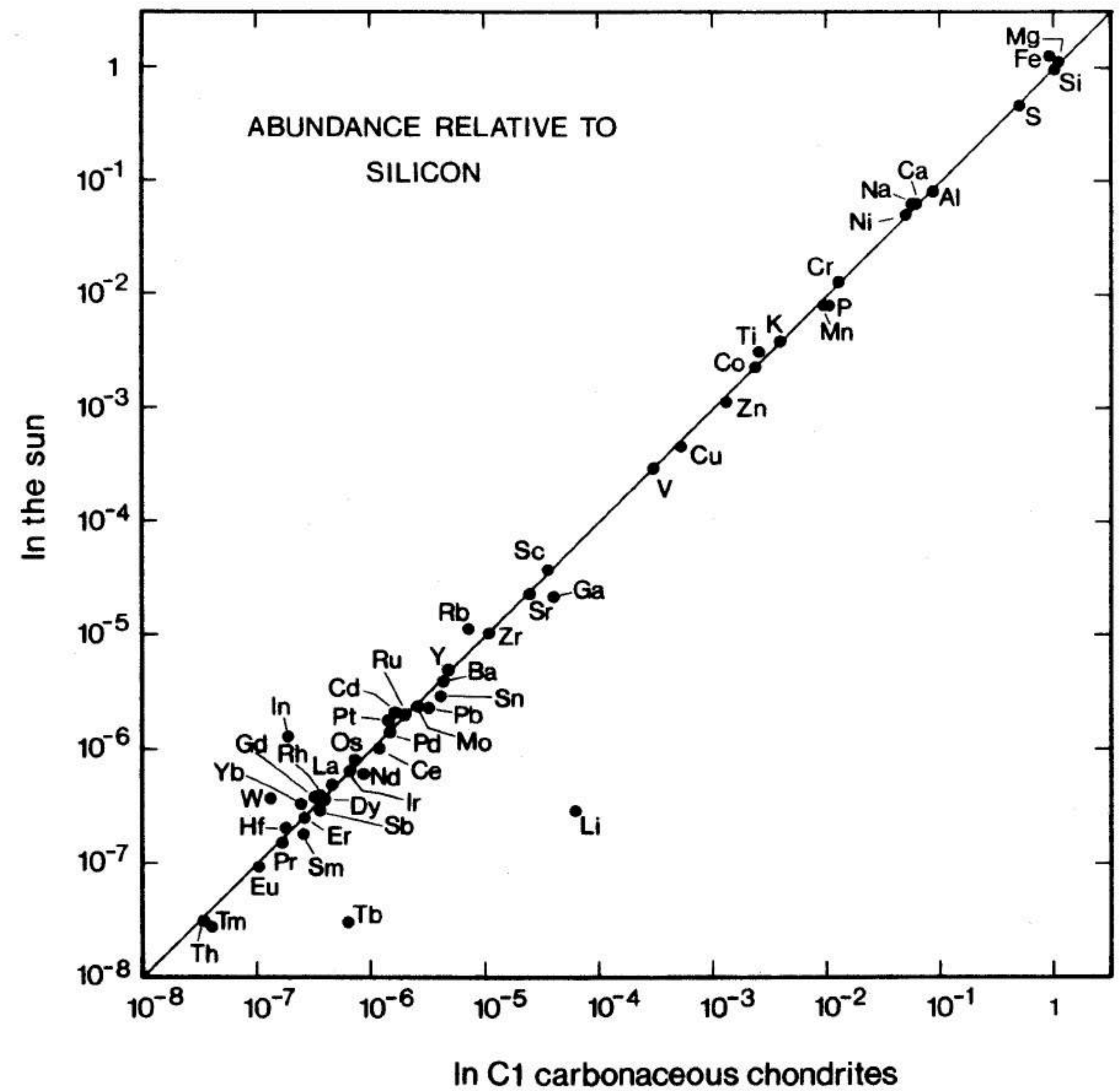
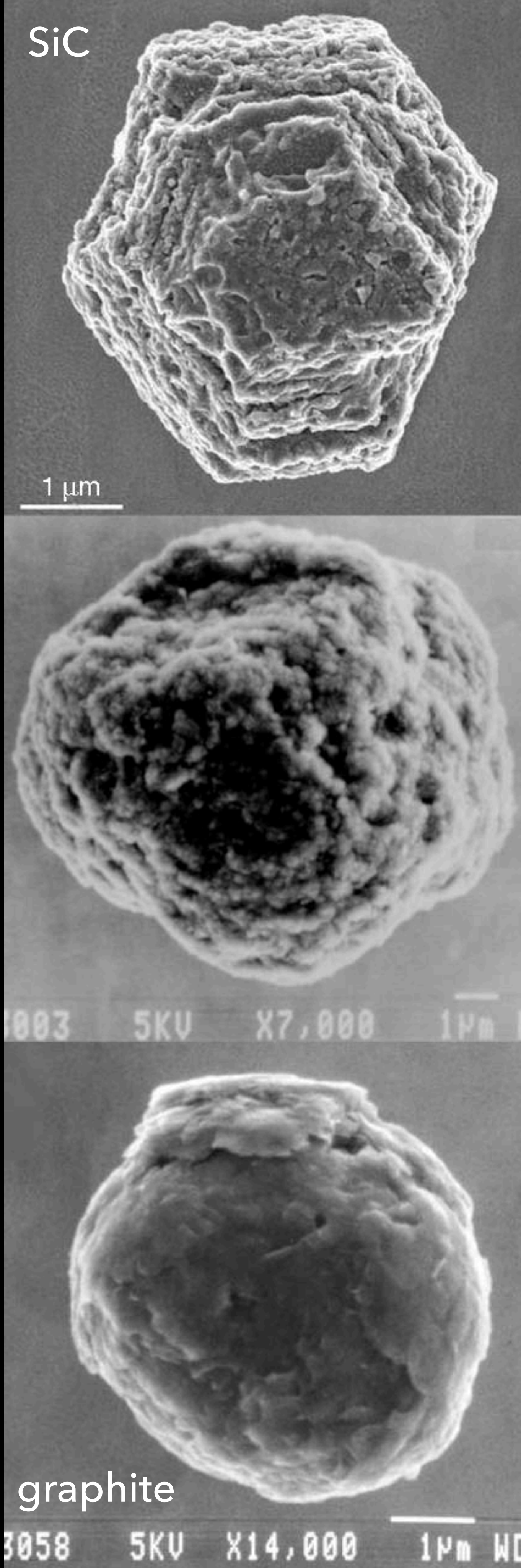


Figure 3 Element/Si ratios of characteristic elements in various groups of chondritic (undifferentiated) meteorites normalized to respective ratios in CI chondrites. Meteorite groups are arranged in order of decreasing oxygen content. The best match between solar photosphere measurements and meteoritic abundances is with CI chondrites (see text for details).



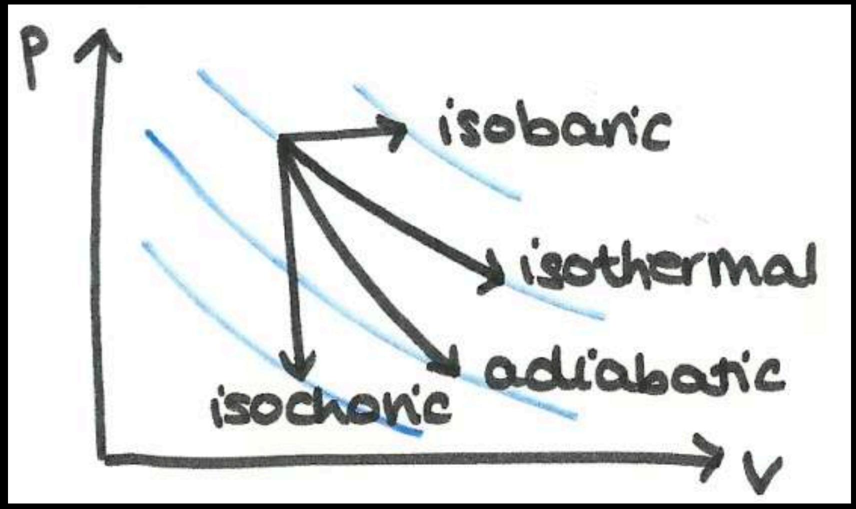
CONDENSATION

- The collapse of an interstellar gas cloud is a violent process and temperatures are high enough to vaporise many solids. Only presolar grains are know for sure to survive:
    - Small refractory grains like nano-diamonds, graphite particles, or silicon carbide (SiC) grains.
- As the newly formed disc cools, new dust grains condense out (probably concurrently) with refractory elements in the inner disc and volatile elements beyond the snow lines.
- We'll assume chemical reactions occur much faster than changes in temperature and density (reasonable assumption in the inner disk where temperatures and densities are high).



CONDENSATION

- In a thermodynamical system, processes will continue spontaneously until the relevant thermodynamical potential is minimised. In equilibrium, e.g.:
  - Helmoltz free energy is minimised for isothermal-isochoric systems: $F = U - TS$
  - Gibbs free energy (also called free enthalpy) is minimised for isothermal-isobaric systems: $G = F + PV = (U - TS) + PV = H - TS$ (as opposed to enthalpy $H = U + PV$)
- For now, let us assume chemical reactions occur in isothermal-isobaric conditions at thermodynamical equilibrium.



CONDENSATION

- Using the first law of thermodynamics ($dU = \delta Q - PdV$):

$$G = H - TS \quad \longrightarrow \quad dG = dH - TdS - SdT$$
$$H = U + PV \quad \longrightarrow \quad dH = dU + PdV + VdP = \delta Q + VdP$$

- For reversible processes (where entropy is $S = \delta Q_{\mathrm{rev}}/T$):

$$dG = \delta Q + VdP - T\left(\frac{\delta Q}{T}\right) - SdT = VdP - SdT$$

- In equilibrium, we can assume $dG = 0$ and the potential is defined to within a constant. Useful to define standard conditions to be used as a reference point
  - Standard conditions are generally set to:

$$T_0 = 298\ \mathrm{K} \qquad P_0 = 1\ \mathrm{atm}$$

CONDENSATION: EXAMPLE

- To illustrate this concept, consider the change in Gibbs free energy of the simple reaction:

$$H_2 + \frac{1}{2}O_2 \rightarrow H_2O$$

- The change in Gibbs free energy at standard conditions (denoted by double subscripts, ΔG_{00}):

$$\begin{aligned}\Delta G_{00} &= G_{00}(H_2O) - G_{00}(H_2) - \frac{1}{2}G_{00}(O_2)\\ &= (-258.8) - (0) - \frac{1}{2}(0) = -258.8\frac{\text{kJ}}{\text{mole}}\end{aligned}$$

- By convention, the Gibbs free energy of the most stable form of a substance is taken to be zero. A negative Gibbs free energy means the reaction is exergonic (net release of free energy) and thus a favoured reaction (spontaneous).

CONDENSATION

- Disc conditions are very different to the standard values. To approximate the Gibbs free energy for different conditions, we consider isothermal and isobaric limits.

- Changes at constant temperature ($dT = 0$):

$$dG = VdP = \left(\frac{nRT}{P}\right) dP \quad \longrightarrow \quad G(P,T) - G_0(T) = nRT \ln\left(\frac{P}{P_0}\right)$$

- If the reaction involves N components, each with different concentrations n_i

$$\Delta G(P,T) - \Delta G_0(T) = \Delta \sum_i RTn_i \ln\left(\frac{P_i}{P_0}\right)$$

- Where P_i is the partial pressure of component i and Δ represents the difference before and after the chemical reaction. Importantly, at equilibrium $\Delta G(P,T) = 0$.

CONDENSATION

- Changes at constant pressure ($dP = 0$):

$$dG = -SdT \quad \longrightarrow \quad G(P,T) - G_0(P) = -\int_{T_0}^{T} S(T)dT$$

- Integrating our earlier definition for entropy:

$$\int dS = \int \frac{\delta Q}{T} \quad \longrightarrow \quad S(T) - S_0 = \int_{T_0}^{T} c_P \frac{dT}{T} = c_P \ln\left(\frac{T}{T_0}\right)$$

- Inserting this above and integrating again over T gives:

$$\Delta G(P,T) - \Delta G_0(P) = -\Delta S_0(T - T_0) - \Delta c_P \left[T \ln\left(\frac{T}{T_0}\right) - (T - T_0)\right]$$

- Combining the results from both limits, gives us a way to approximate the Gibbs free energy at arbitrary T and P using standard conditions computed in the lab on Earth:

$$\Delta G(P,T) = 0 = \Delta G_{00} - \Delta G_0(T) - \Delta G_0(P)$$

CONDENSATION: DISSOCIATION OF H_2

separates into two H

- A more realistic (and relevant) reaction: $H_2 \rightarrow H + H$

$$-\frac{\Delta G_0(T)}{RT} = \ln K_P(T) = \Delta \sum_i n_i \ln\left(\frac{P_i}{P_0}\right) = \ln \frac{\left(\frac{P_H}{P_0}\right)^2}{\left(\frac{P_{H_2}}{P_0}\right)} = \ln\left(\frac{P_H^2}{P_{H_2}P_0}\right)$$

- To deal with the partial pressures, it is convenient to define the dissociated fraction α, such that $\alpha = [0, 1]$ refer to pure $[H_2, H]$, respectively. If n is the number of moles

	H_2	H	total
# of moles	$(1-\alpha)n$	$2\alpha n$	$(1+\alpha)n$
molar fraction	$(1-\alpha)/(1+\alpha)$	$2\alpha/(1+\alpha)$	1
partial pressure	$(1-\alpha)P_{tot}/(1+\alpha)$	$2\alpha P_{tot}/(1+\alpha)$	P_{tot}

CONDENSATION: DISSOCIATION OF H_2

- Assuming our disc model will provide P_{tot}, we substitute in the partial pressures to obtain the reaction rate

$$K_P(T) = \frac{\frac{4\alpha^2}{(1+\alpha)^2} P_{\mathrm{tot}}^2}{\left(\frac{1-\alpha}{1+\alpha}\right) P_{\mathrm{tot}} P_0} = \frac{4\alpha^2}{1-\alpha^2} \frac{P_{\mathrm{tot}}}{P_0}$$

- Or solving for the dissociated fraction: $\alpha = \left(\frac{4P_{\mathrm{tot}}}{P_0 K_P(T)} + 1 \right)^{-\frac{1}{2}}$

- Meanwhile the entropy at constant pressure is:

$$\Delta S(T) = \Delta S_0 + \Delta c_P \ln\left(\frac{T}{298\ \mathrm{K}}\right)$$

$$= 2S_0^{\mathrm{H}} + 2c_P^{\mathrm{H}} \ln\left(\frac{T}{298\ \mathrm{K}}\right) - S_0^{\mathrm{H}_2} - c_P^{\mathrm{H}_2} \ln\left(\frac{T}{298\ \mathrm{K}}\right)$$

CONDENSATION: DISSOCIATION OF H_2

- The specific heats we get from an ideal gas

$$c_P^{\mathrm{H}} = \frac{f+2}{2}R = \frac{5}{2}R \qquad c_P^{\mathrm{H}_2} = \frac{7}{2}R \qquad \Delta c_P = 2c_P^{\mathrm{H}} - c_P^{\mathrm{H}_2} = \frac{3}{2}R$$

- Lookup tables provide the numerical values we need:

$$S_0^{\mathrm{H}} = 114.72\,\frac{\mathrm{J}}{\mathrm{mole\ K}} \qquad S_0^{\mathrm{H}} = 114.72\,\frac{\mathrm{J}}{\mathrm{mole\ K}} \qquad \Delta S_0 = 98.76\,\frac{\mathrm{J}}{\mathrm{mole\ K}}$$

$$\Delta G_{00} = 2 \times 2.0328 \times 10^3 - 0 = 4.0356 \times 10^5\ \mathrm{J/mole}$$

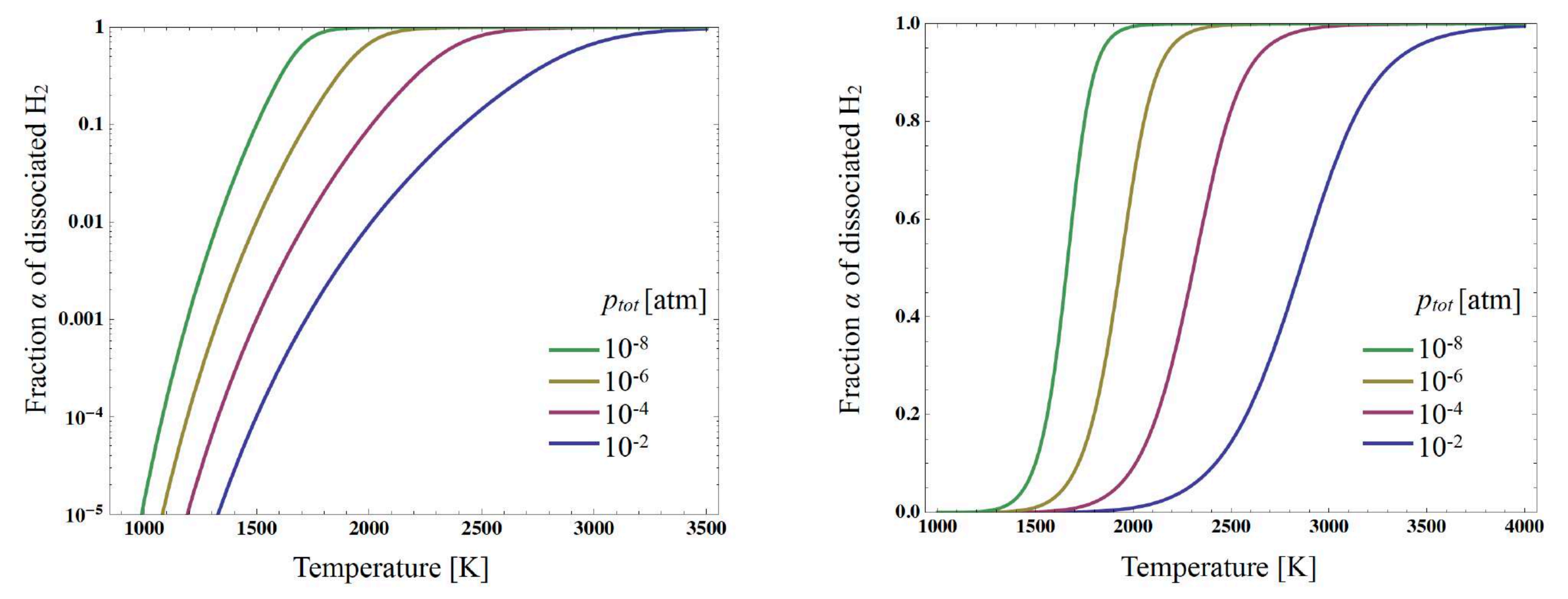
- Plugging all of these values into our final equation

$$\Delta G_0(T) = \Delta G_{00} - \Delta S_0(T - T_0) - \Delta c_P \left[T \ln\left(\frac{T}{T_0}\right) - (T - T_0) \right]$$

$$= 4.0356 \times 10^5 - 98.76(T - 298) - \frac{3}{2}\left[T \ln\left(\frac{T}{298}\right) - (T - 298) \right]$$

- We can then calculate $K_P(T) = \mathrm{e}^{-\frac{\Delta G_0(T)}{RT}}$, and finally $\alpha(P_{\mathrm{tot}}, T)$.

CONDENSATION: DISSOCIATION OF H_2



- A high total pressure inhibits dissociation.
- Dissociation begins suddenly and is a strong function of T and P.
- For $T \gtrsim 3500$ K, the gas is atomic (only in the inner disc).
  For $T \lesssim 1000$ K, the gas is molecular (majority of the disc is H_2).
- Very idealised…remember we made a lot of assumptions.

CONDENSATION: IRON EXAMPLE

- At equilibrium, for $\mathrm{Fe_g} \rightarrow \mathrm{Fe_s}$, we set the abundances and the partial pressure of the solid to unity:

$$-\frac{\Delta G_0(T)}{RT} = \ln K_P(T) = \Delta \sum_i n_i \ln\left(\frac{P_i}{P_0}\right) = \ln\left(\frac{P_{\mathrm{Fe_s}}/P_0}{P_{\mathrm{Fe_g}}/P_0}\right) = -\ln P_{\mathrm{Fe_g}}$$

- As before, we look up numerical values in tables

$$S_{\mathrm{Fe_s}} = 27.06 + 25.10 \ln\left(\frac{T}{298}\right) \qquad \Delta S_{\mathrm{Fe_g}} = -153.42 - 0.58 \ln\left(\frac{T}{298}\right)$$

$$S_{\mathrm{Fe_g}} = 180.49 + 25.68 \ln\left(\frac{T}{298}\right) \qquad \Delta G_{00} = -3.698 \times 10^5 \ \mathrm{J/mole}$$

$$\Delta G_0(T) = -3.698 \times 10^5 + 153.42(T - 298) - 0.58\left[T \ln\left(\frac{T}{298}\right) - (T - 298)\right]$$

- For $T \approx T_0$, we get $\Delta G_0 \approx -3.698 \times 10^5$ and $P_{\mathrm{Fe_g}} \propto \mathrm{e}^{-\frac{4473}{T}}$

CONDENSATION: IRON EXAMPLE

- Assume that $P_{\rm tot} \approx P_{\rm H_2} + P_{\rm He}$ remains constant (i.e. not affected by vaporised Fe). The $P_{\rm Fe_g}$ follows from abundance considerations:

$$\frac{P_i}{P_{\rm tot}} = \frac{n_i}{n_{\rm tot}} = X_i \approx \frac{n_i}{n_{\rm H} + n_{\rm He}} \qquad N(\rm el) \equiv \frac{n(\rm el)}{n(\rm Si)} \times 10^6$$

- On the cosmochemical scale, atomic abundances are normalised to the number of Si atoms: $\log_{10} N(\rm Si) = 6$. Assuming H is in molecular form and using standard abundances for the solar nebula:

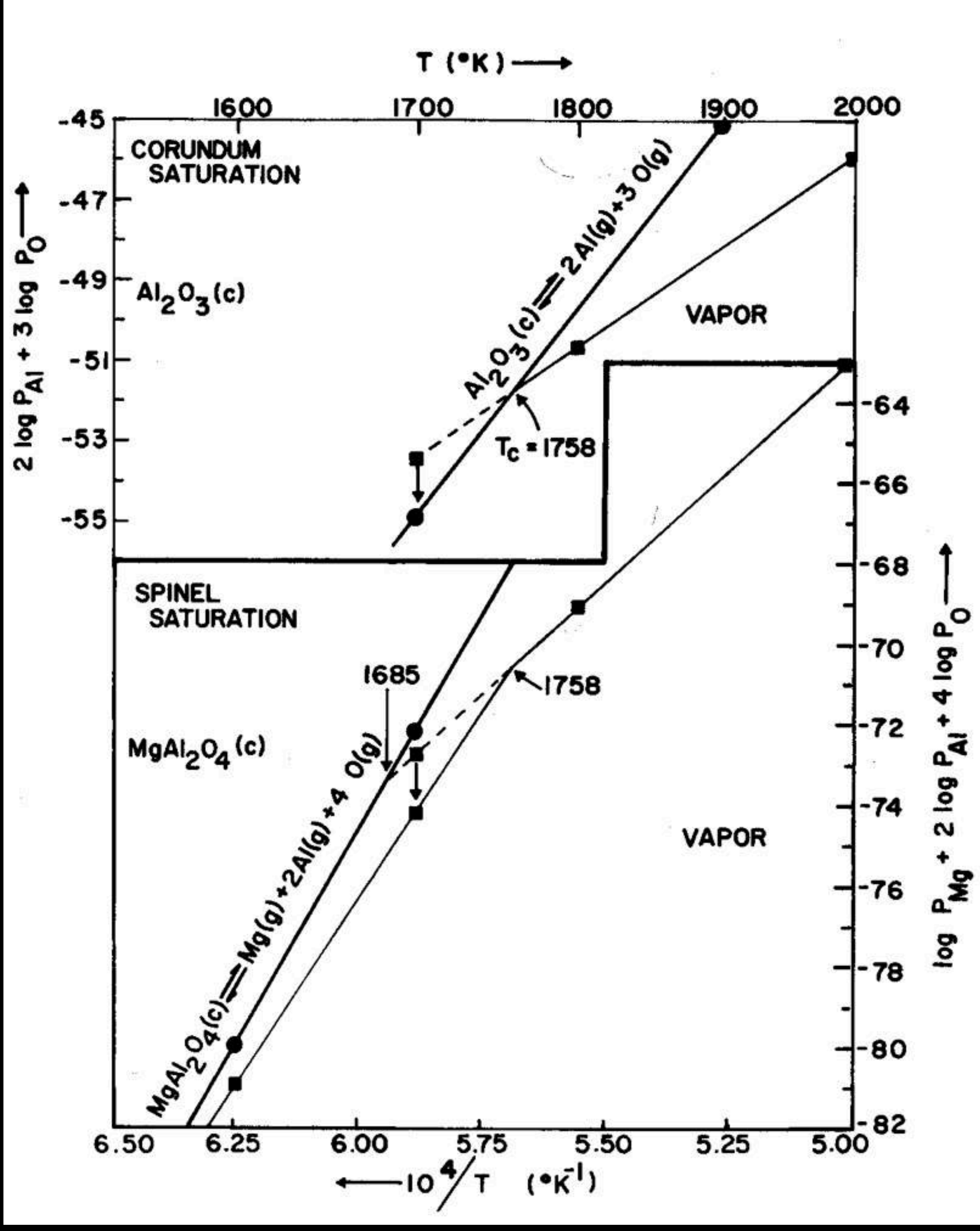
$\log_{10} N(\rm Fe) = 5.95, \qquad \log_{10} N(\rm H) = 10.45, \qquad \log_{10} N(\rm He) = 9.45$

$$P_{\rm Fe} = P_{\rm tot} \left[\frac{N(\rm Fe)}{0.5N(\rm H) + N(\rm He)} \right] = 5.31 \times 10^{-5} P_{\rm tot}$$

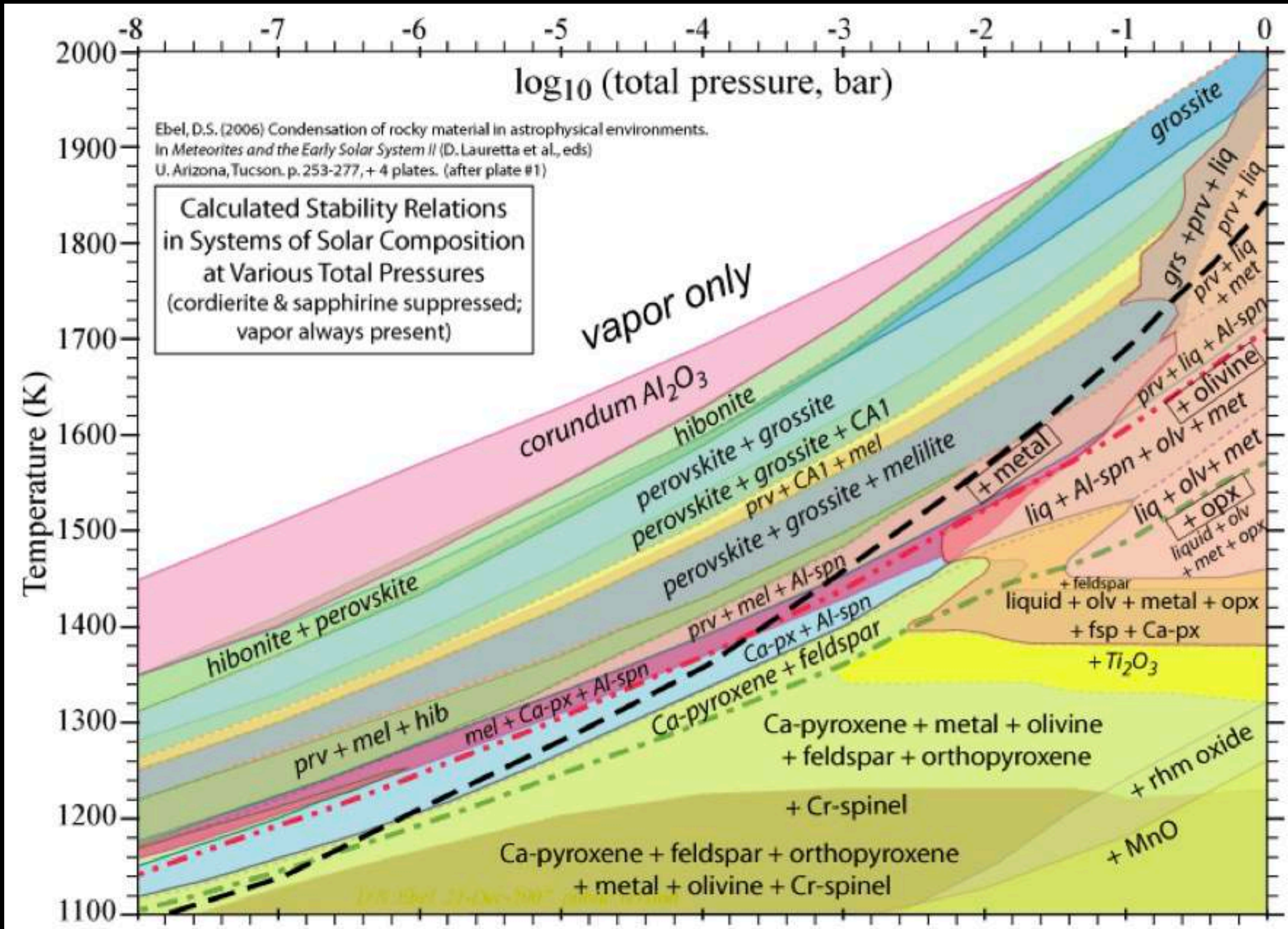
- This partial pressure plots as a horizontal line in the diagram. The intersection yields the condensation temperature of Fe as condensation occurs when the vapour pressure is equal the partial pressure.

CONDENSATION: FULL SEQUENCE

- In more detailed models, the vapour phase is not a horizontal line (relative abundances depend on T and P).
- Normally, spinel would condense at $T = 1685$ K, but corundum condenses first and removes Al and O, causing the slope of the partial pressure to change.
- Condensation for spinel now happens at $T = 1500$ K.



CONDENSATION: FULL SEQUENCE



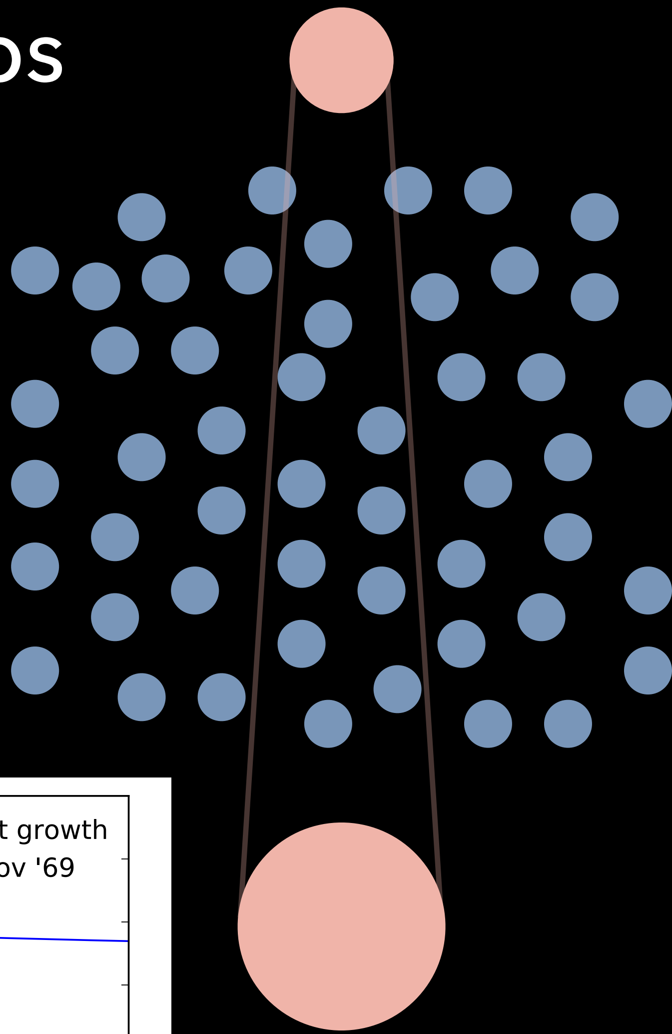
FROM UNIVERSE

TO PLANETS

LECTURE 3.2: GROWTH/FRAGMENTATION

COAGULATION

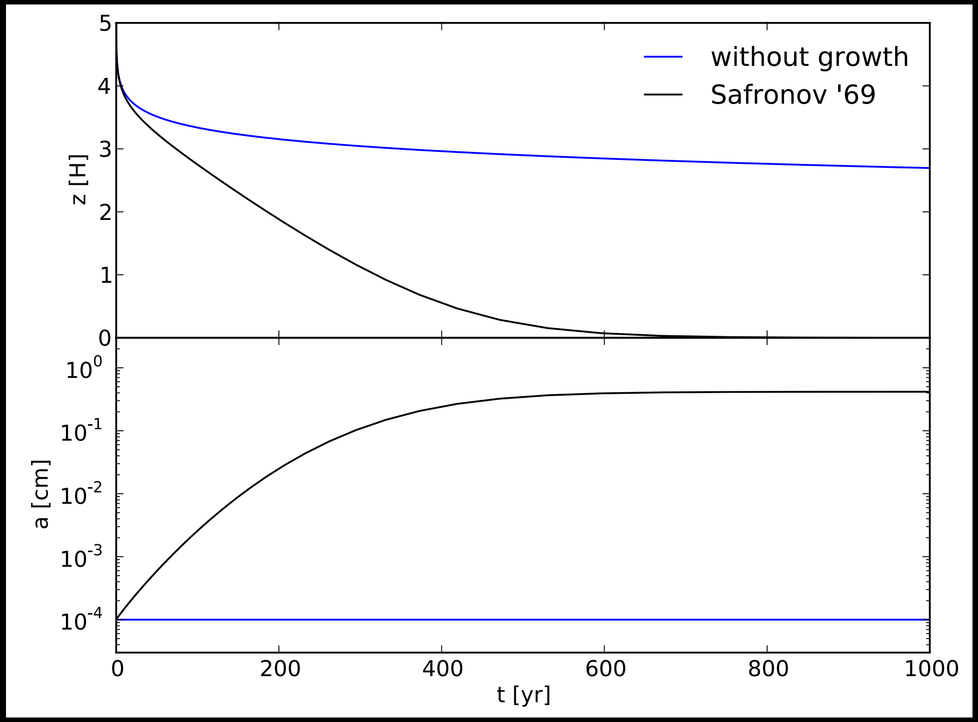
- Vertical settling timescale is much faster than the radial drift timescale. Simple model: the dust sweeps up grains as it settles at terminal velocity.



$$dm = \underbrace{\pi a^2 |v_z| dt}_{\text{volume}} \times \underbrace{\rho_g \varepsilon}_{\text{dust density}}$$

$$\frac{da}{dt} = \frac{\varepsilon \Omega_K^2}{4 v_{th}} z a \qquad v_d^z = -z\Omega_K \mathrm{St}(a, z)$$

- Solving this numerically:

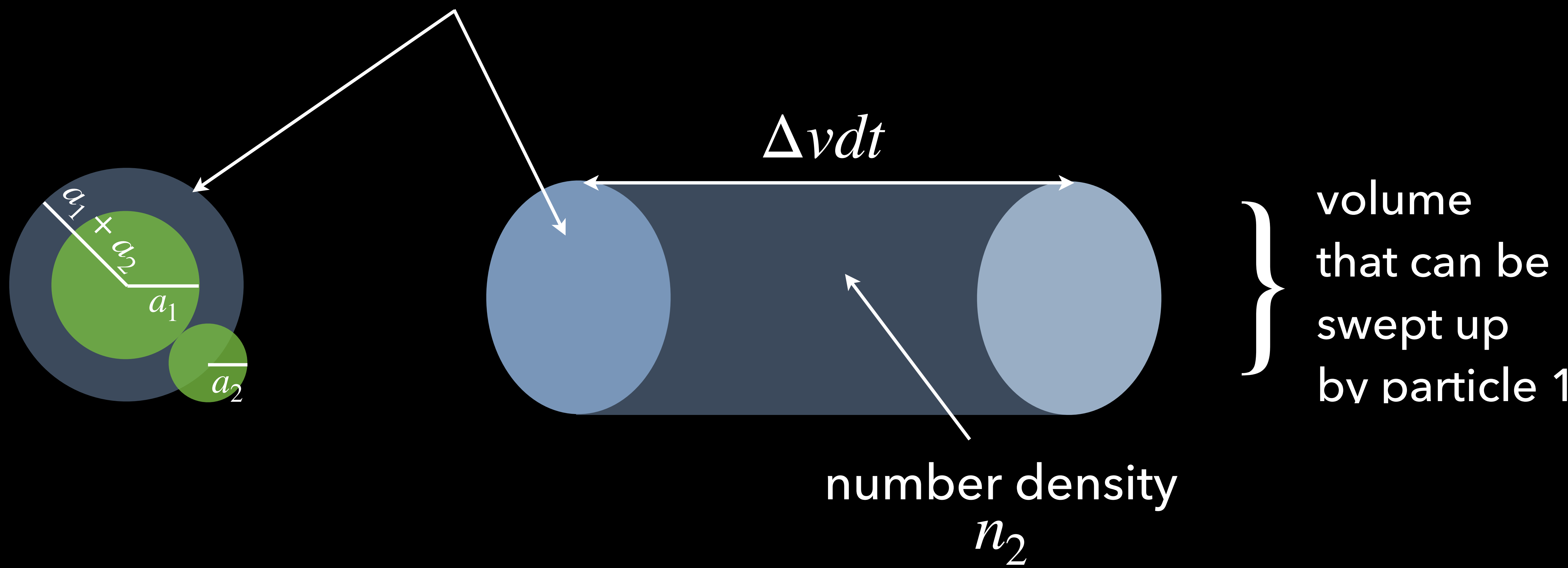


- Differences in the condensation sequence can fractionate the disc.

COAGULATION

cross section

$$\sigma = \pi(a_1 + a_2)^2$$



- For one particle of m_1: $\dfrac{\#\text{ collisions}}{\text{time}} = \sigma \Delta v n_2$

- But we have n_1 of them: $\dfrac{dn_3}{dt} = \sigma \Delta v n_1 n_2$

- The fraction S that lead to sticking: $\dfrac{dn_3}{dt} = \underbrace{S\sigma\Delta v}_{K\,=\,\text{coagulation kernel}}\ n_1 n_2$

Describes the rate at which particles of size 1 coagulate with particles of size 2.

COAGULATION

- So particles of mass m are produced according to:

Joining particles reduces the # by half

Only pick off collisions that contribute to this mass bin

$$\left.\frac{dn(m)}{dt}\right|^{+} = \frac{1}{2}\int_{m_i}\int_{m_j} K(m_i, m_j)n(m_i)n(m_j)\delta(m_i + m_j, m)\, dm_i dm_j$$

Masses > m do not contribute

$$= \frac{1}{2}\int_0^m K(m', m - m')n(m')n(m - m')\, dm'$$

- But they also get swept up by all other sizes:

$$\left.\frac{dn(m)}{dt}\right|^{-} = n(m)\int_0^\infty K(m, m')n(m')\, dm'$$

COAGULATION

- Together we can track mass changes due to growth:

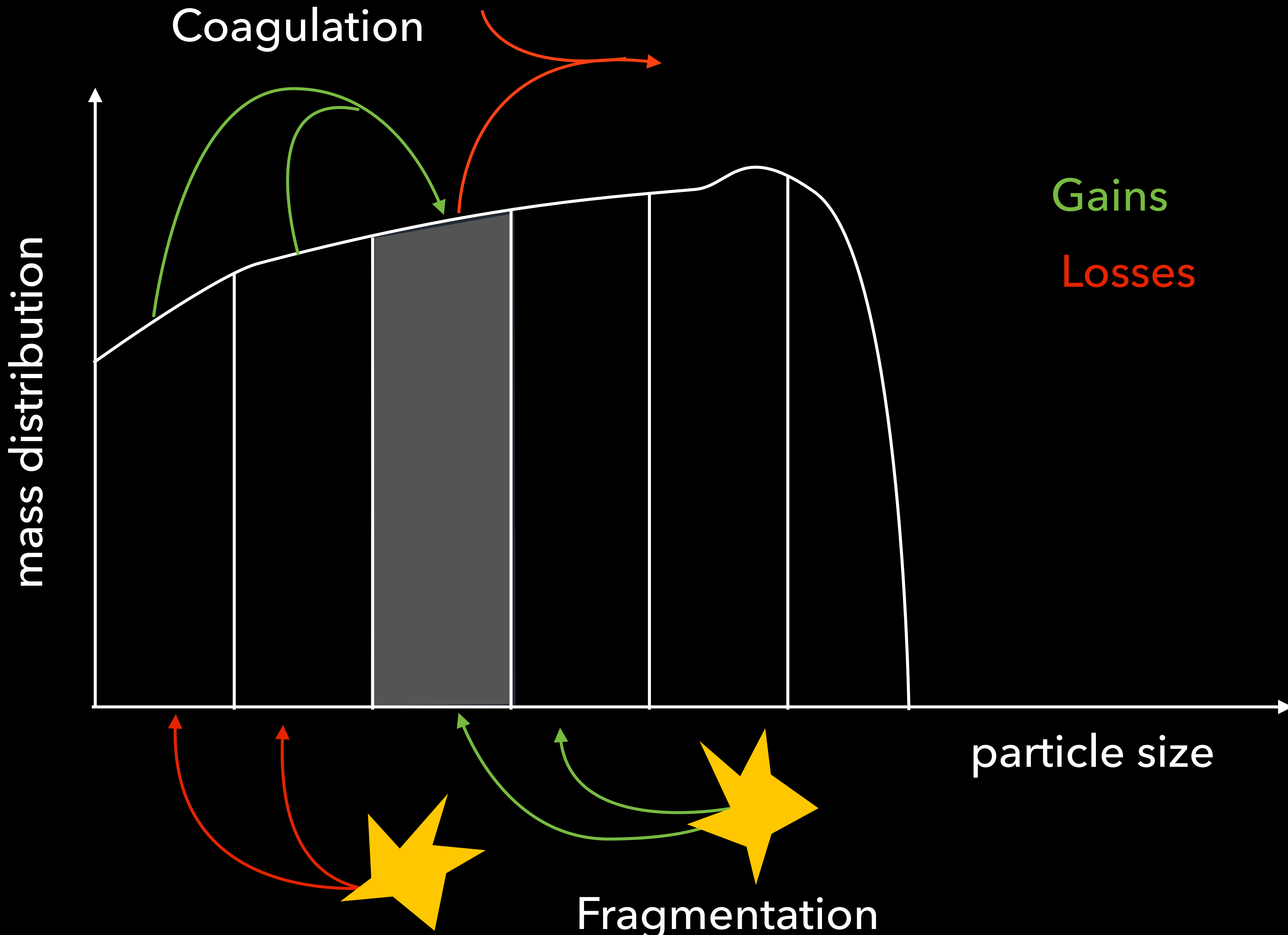
$$\frac{dn(m)}{dt} = \frac{1}{2}\int_0^m K(m', m - m')n(m')n(m - m')\, dm'$$
$$- n(m)\int_0^\infty K(m, m')n(m')\, dm'$$

- More generally, we should consider all types of collisions (sticking, bouncing, fragmentation) and incorporate these into the kernel:

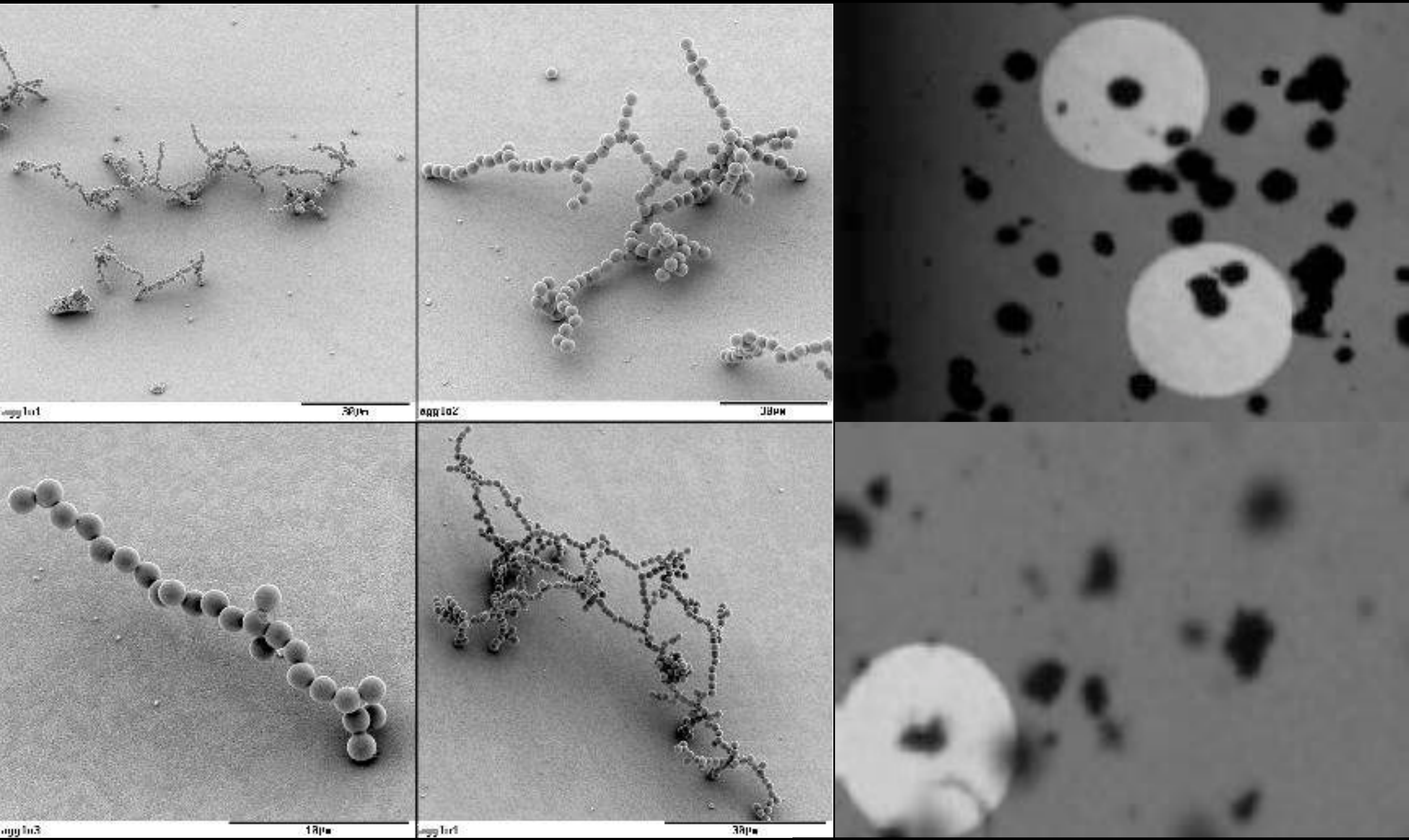
$$\frac{dn(m)}{dt} = \int_0^\infty \int_0^\infty K(m, m_1, m_2)n(m_1)n(m_2)\, dm_1 dm_2$$

- This is only one dimension (mass). We haven't considered porosity, charge, composition…

COAGULATION + FRAGMENTATION



COAGULATION + FRAGMENTATION



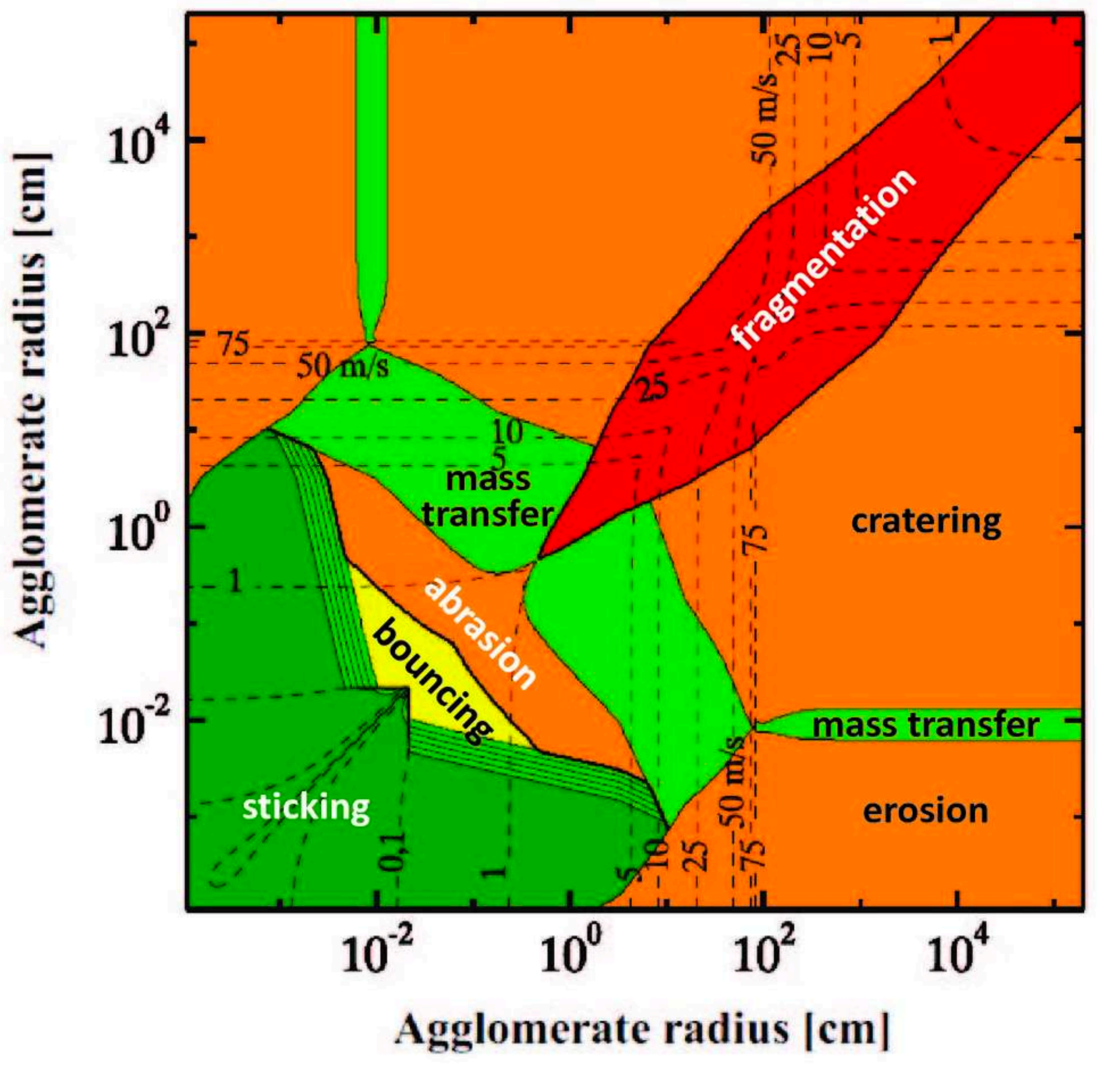
COAGULATION + FRAGMENTATION

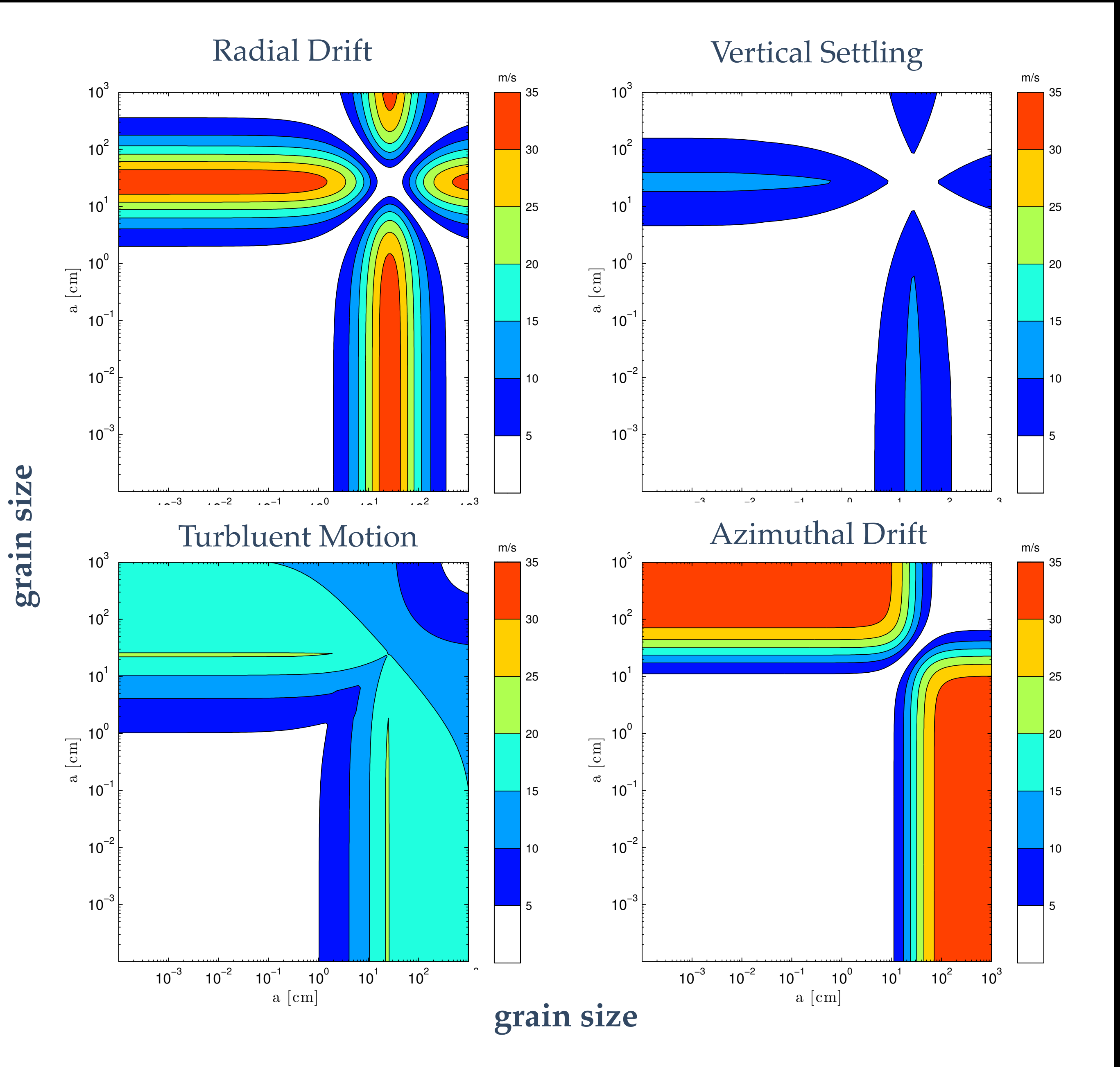
COAGULATION + FRAGMENTATION

ダストの衝突合体成長

Collisional Growth of Dust

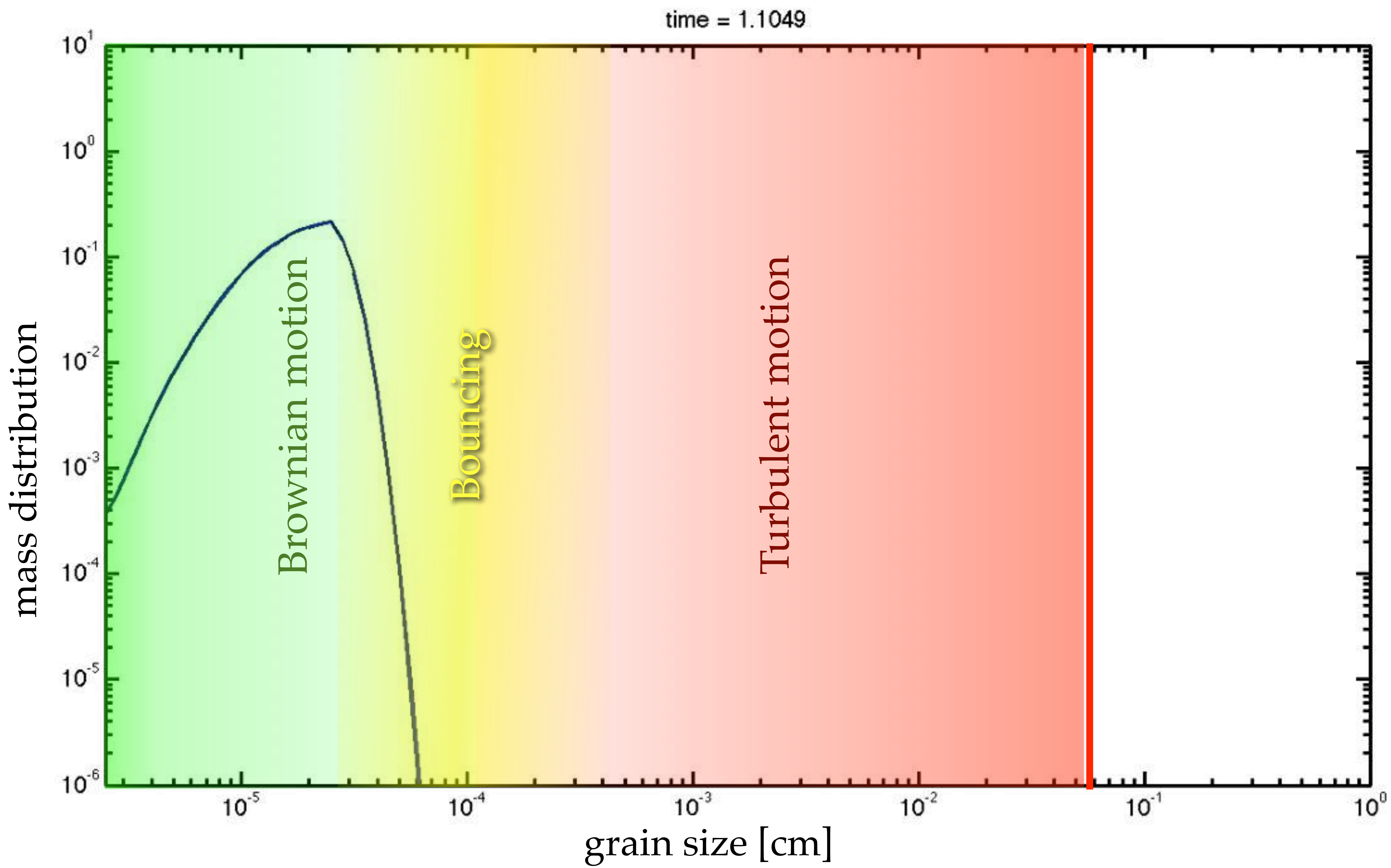
COAGULATION + FRAGMENTATION



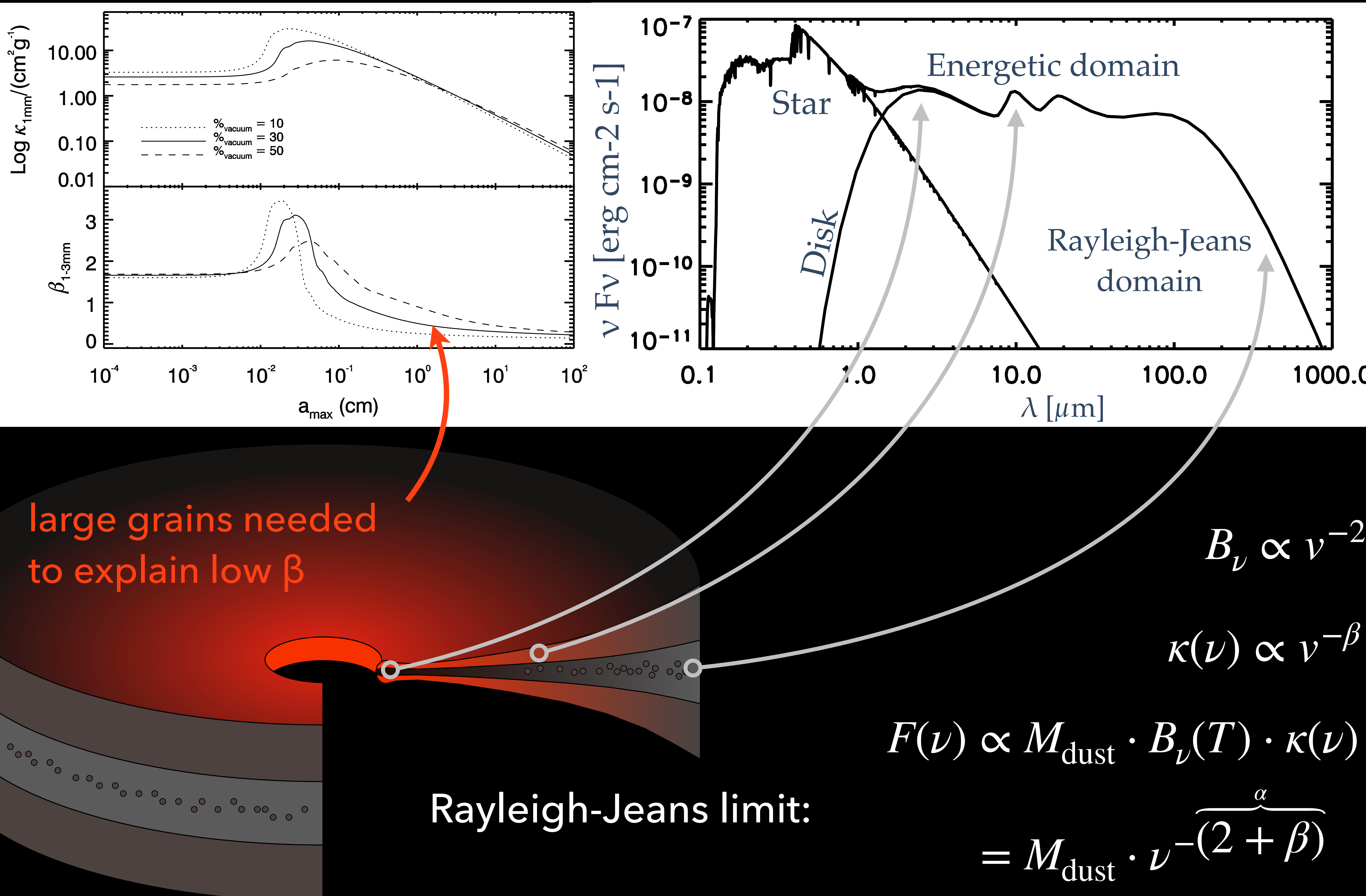


- Small particles are sticky, velocities given by Brownian motion.
- Turbulence and differential motion dominates for larger particles.
- Impact velocities increase with particle size → problem!

COAGULATION + FRAGMENTATION



EVIDENCE OF GRAIN GROWTH



Rayleigh-Jeans limit:

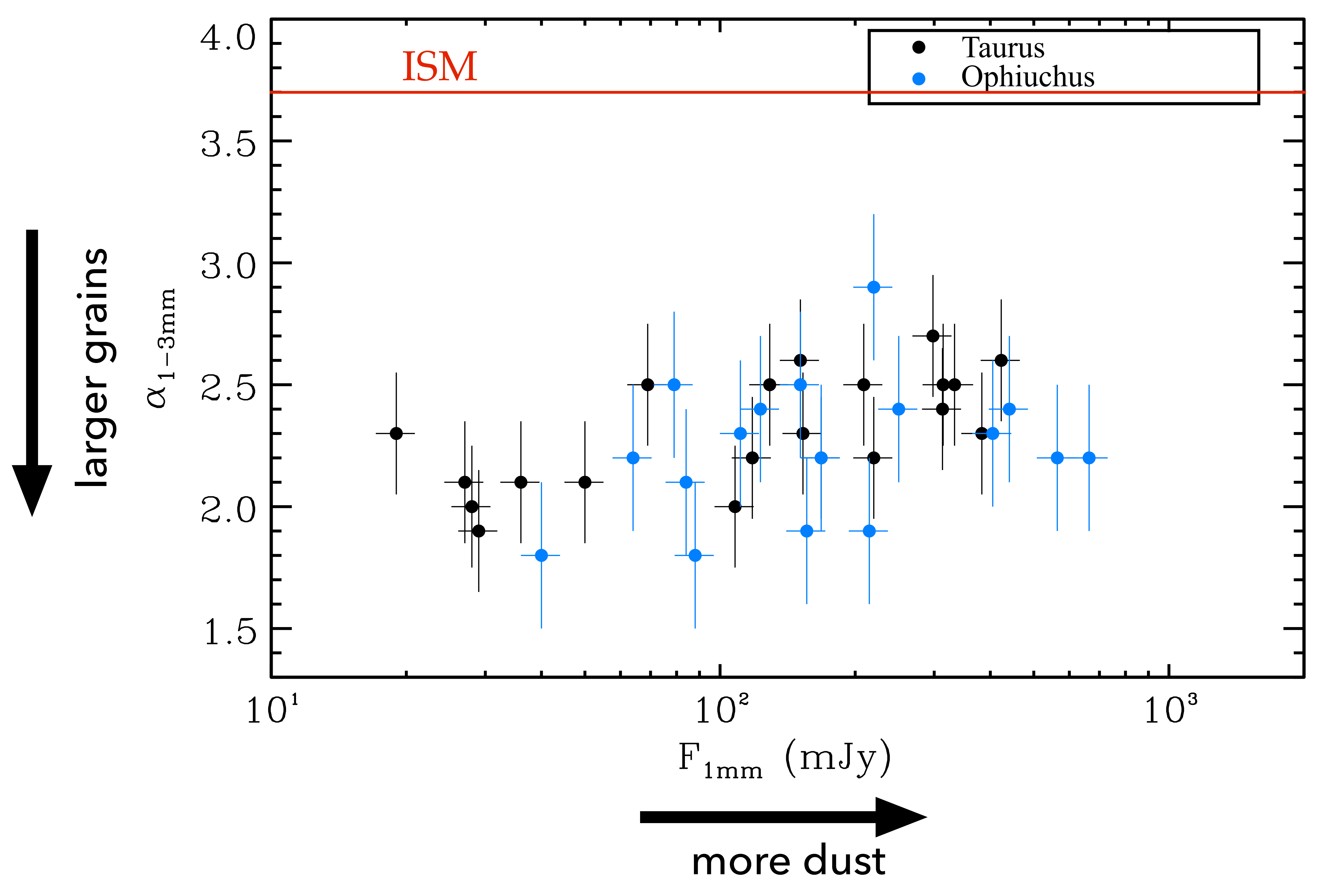
$$B_\nu \propto \nu^{-2}$$

$$\kappa(\nu) \propto \nu^{-\beta}$$

$$F(\nu) \propto M_{\text{dust}} \cdot B_\nu(T) \cdot \kappa(\nu)$$

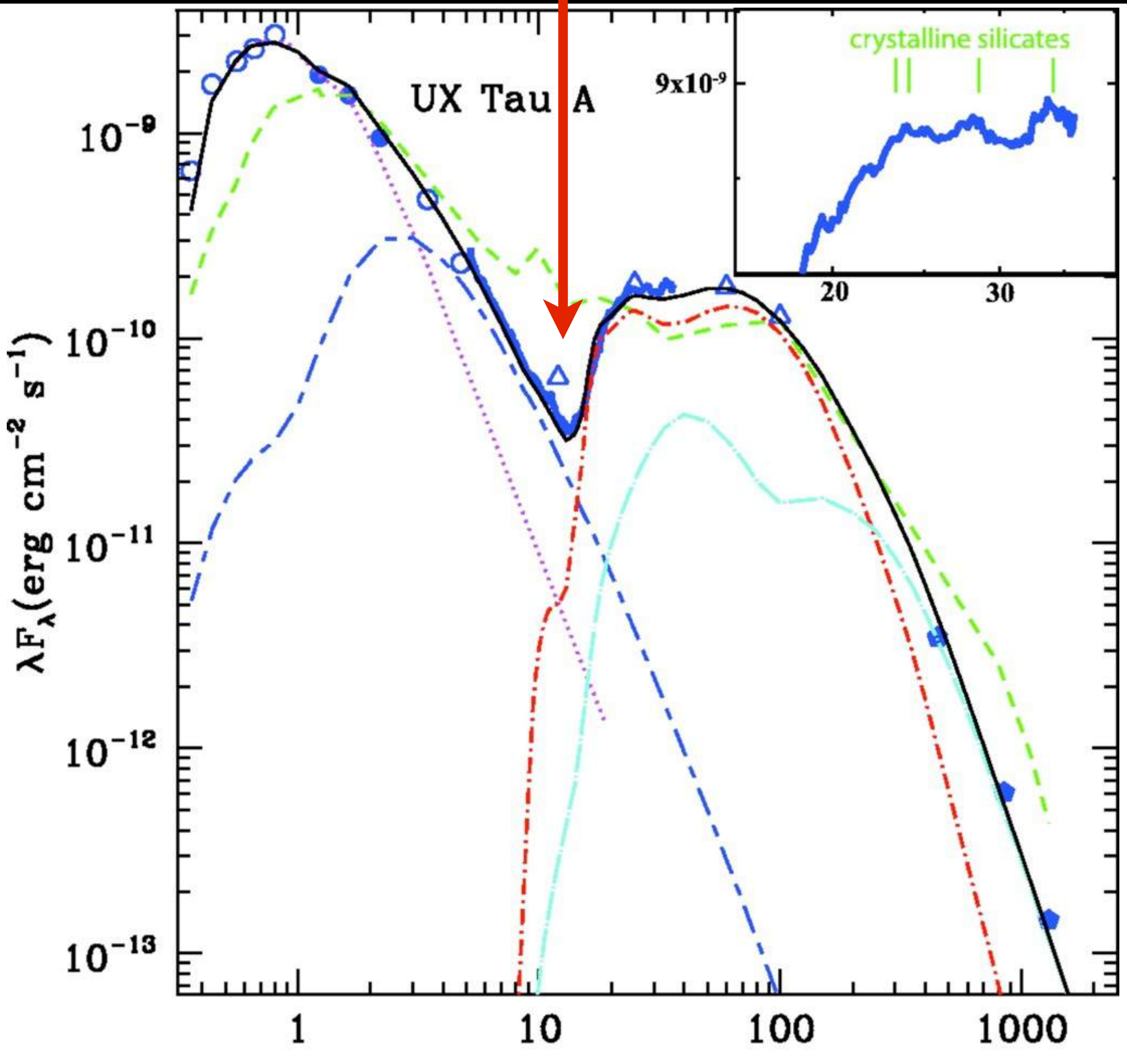
$$= M_{\text{dust}} \cdot \nu^{-\overbrace{(2+\beta)}^{\alpha}}$$

EVIDENCE OF GRAIN GROWTH

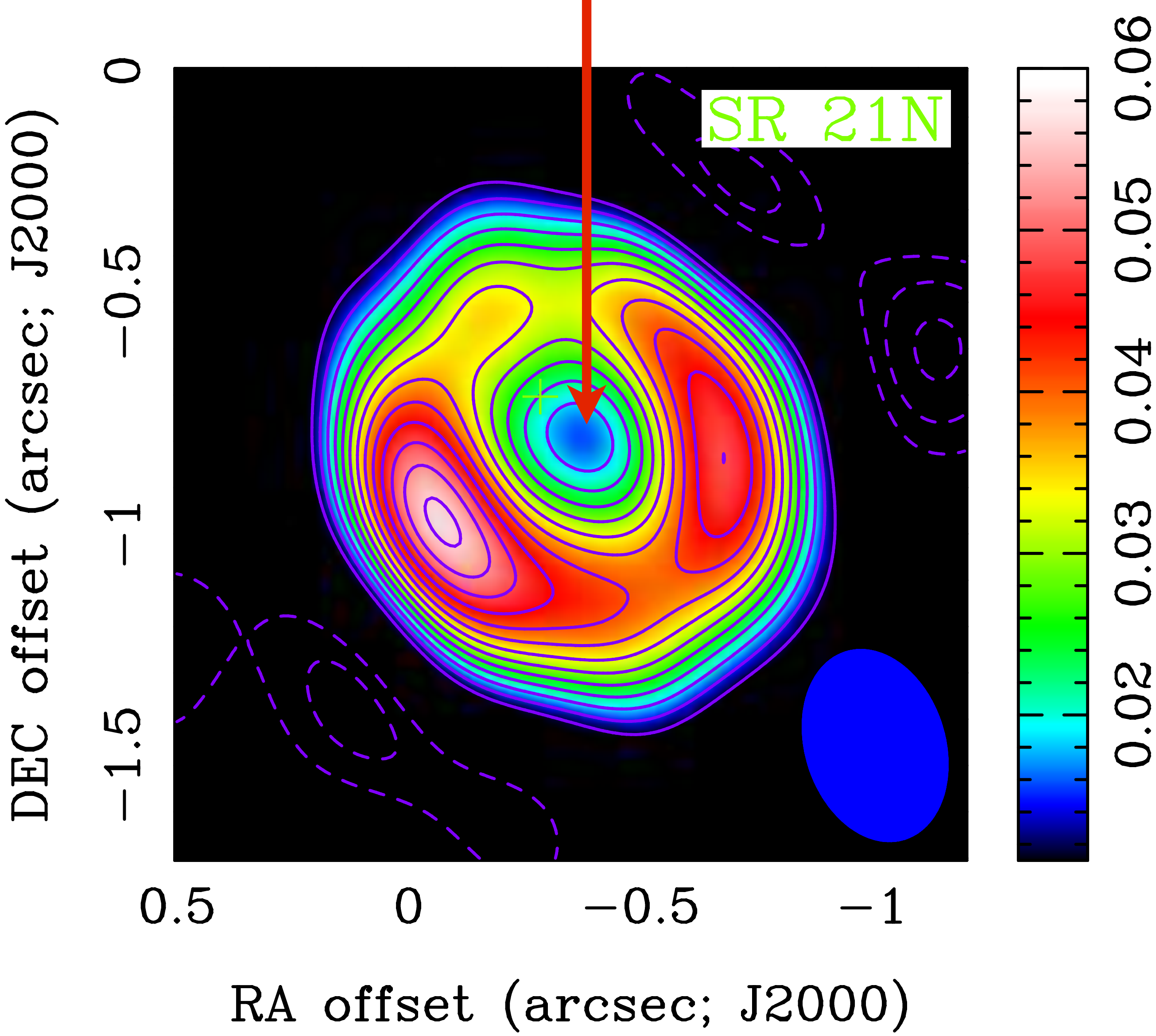


EVIDENCE OF GRAIN GROWTH

Warm dust in inner regions is missing

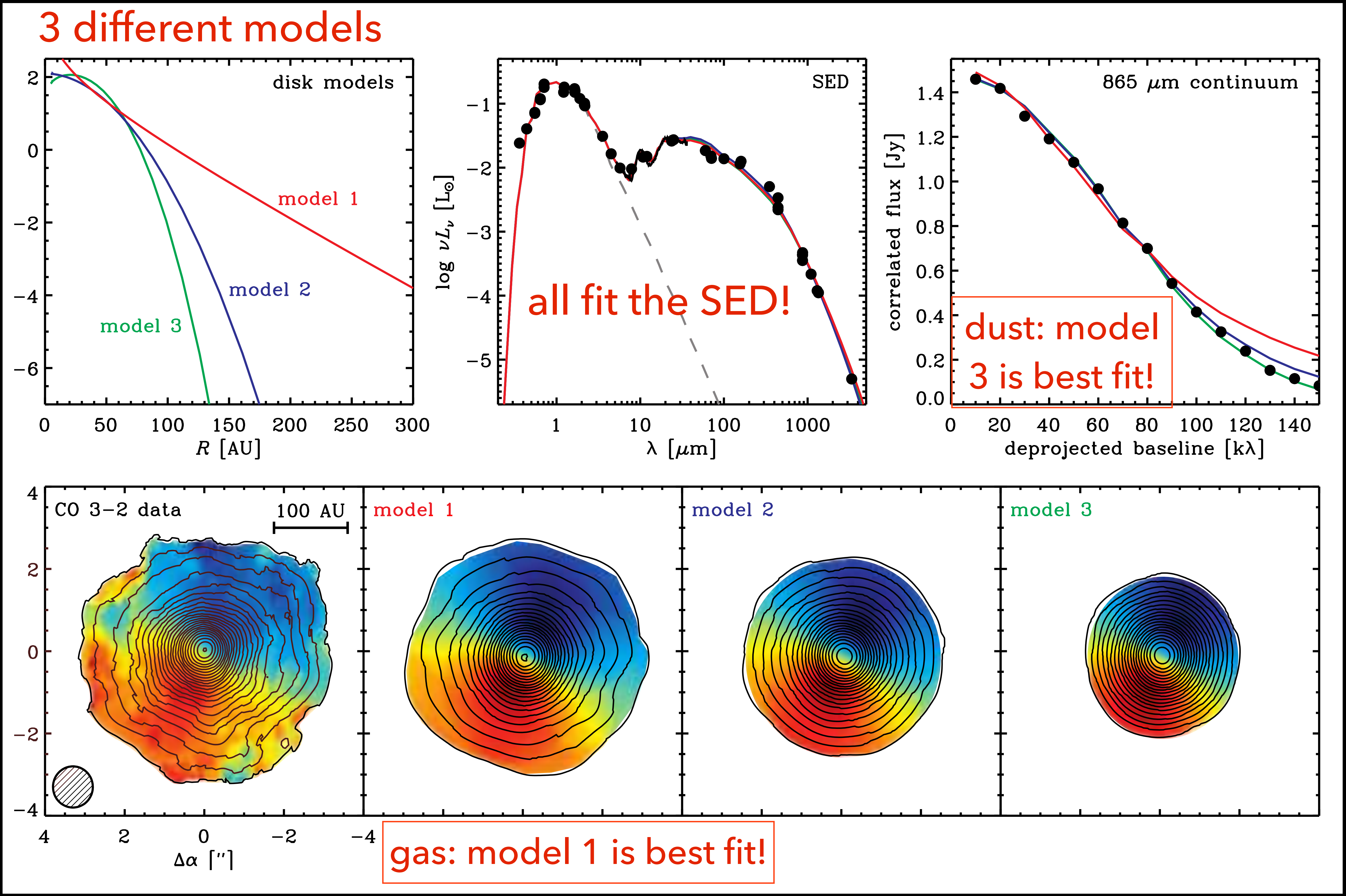


Espaillat et al. 2007

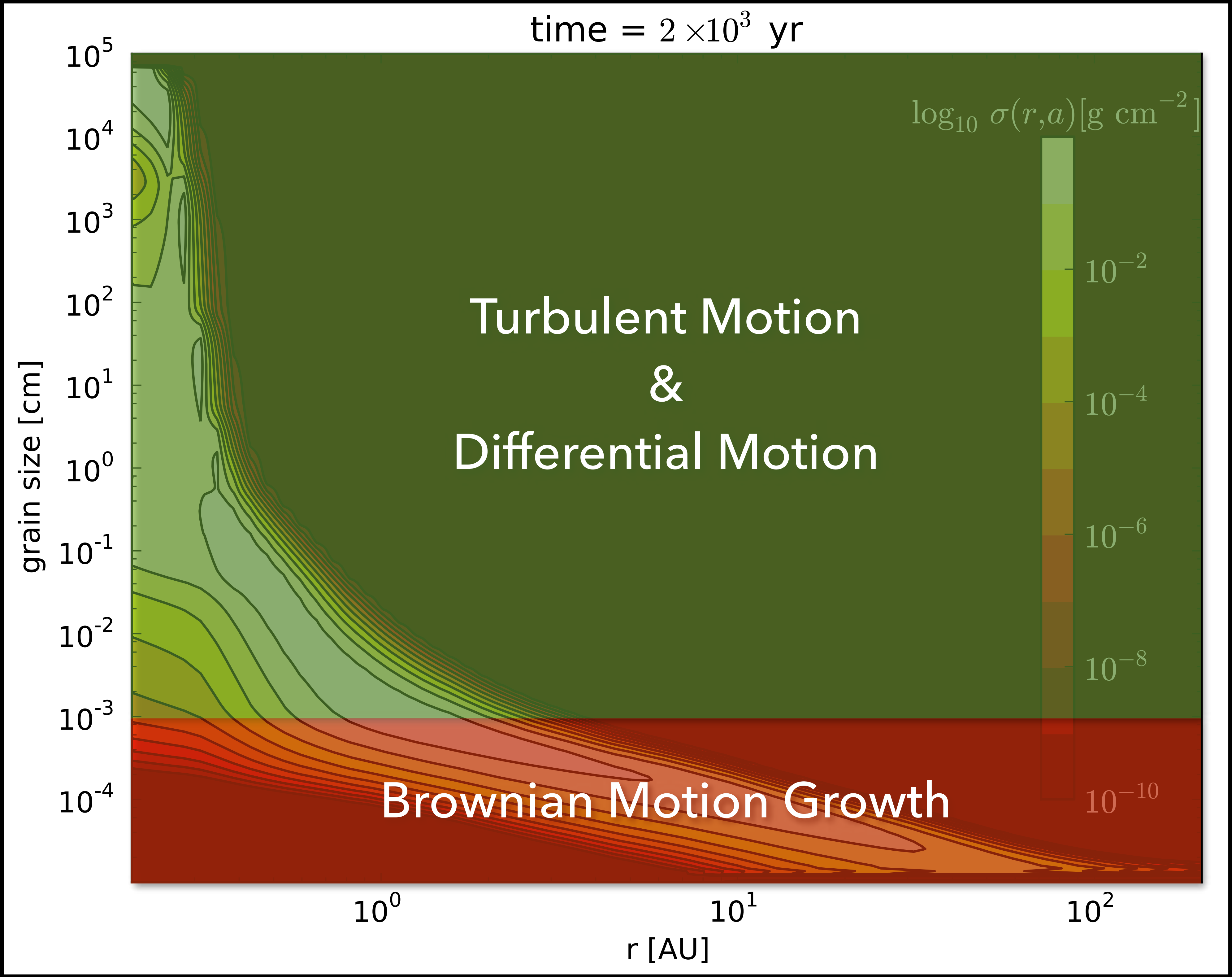


Brown et al. 2009

EVIDENCE OF GRAIN GROWTH



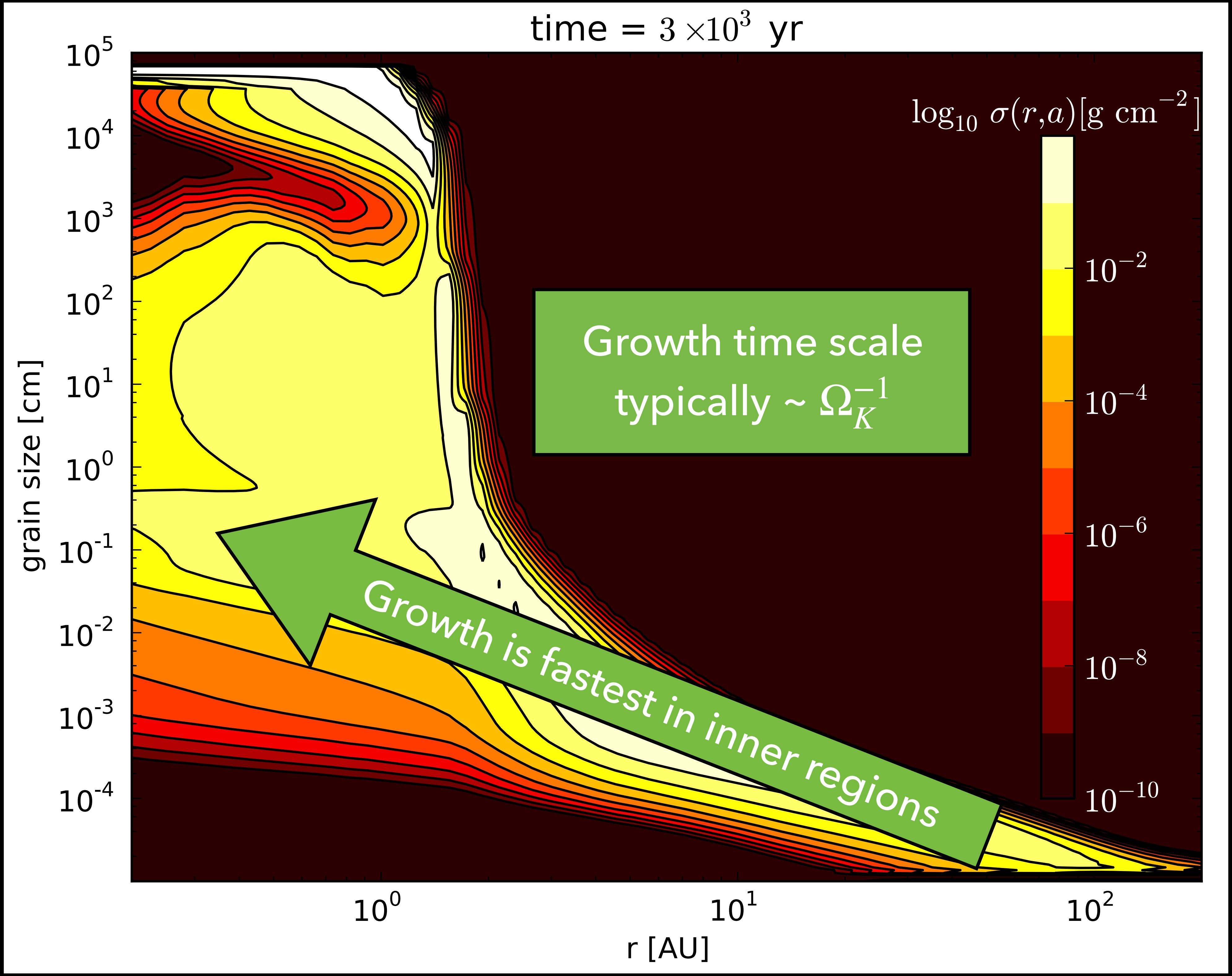
GROWTH BARRIERS



Birnstiel et al. 2010, 2012

GROWTH BARRIERS

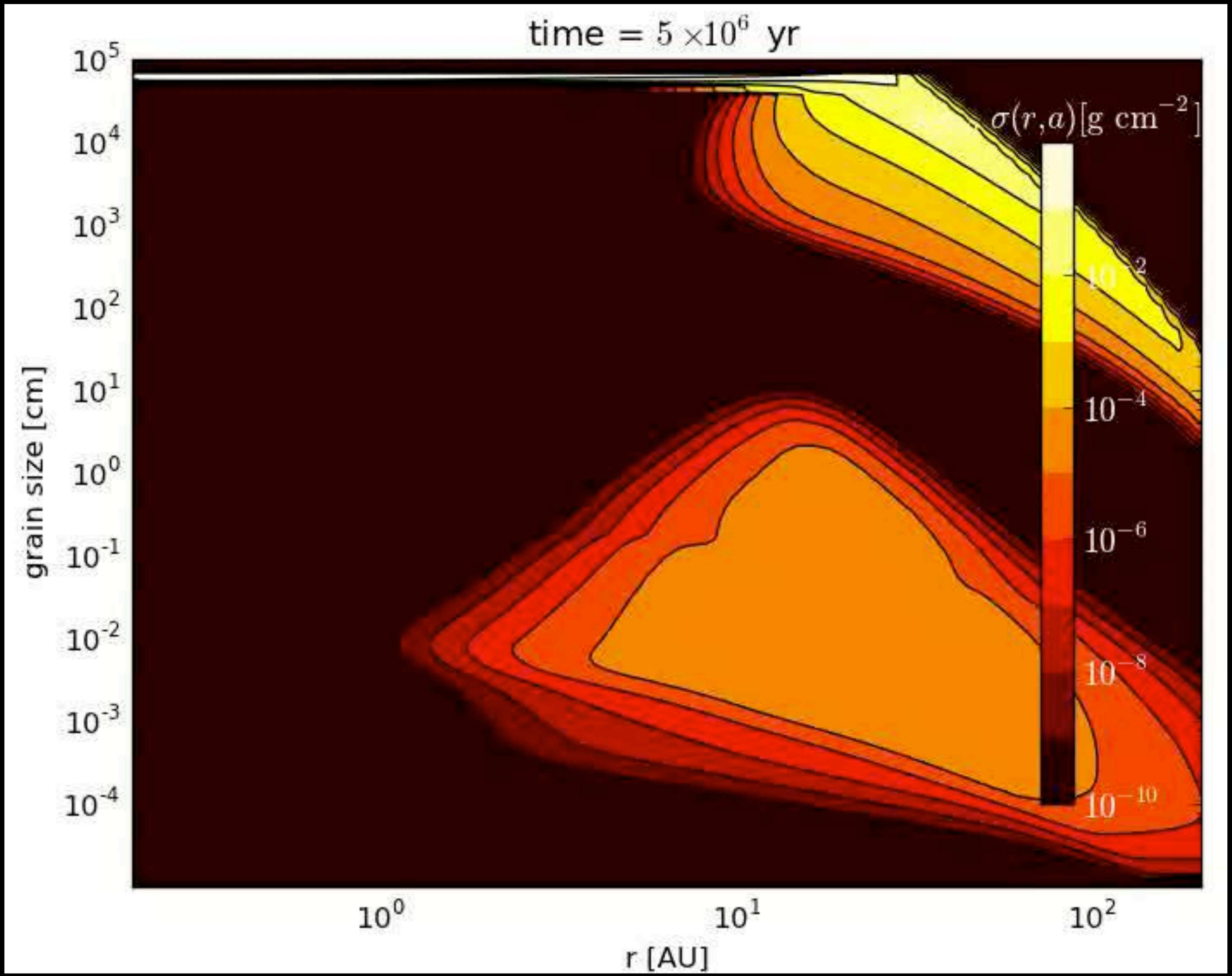
Only grain growth



Birnstiel et al. 2010, 2012

GROWTH BARRIERS

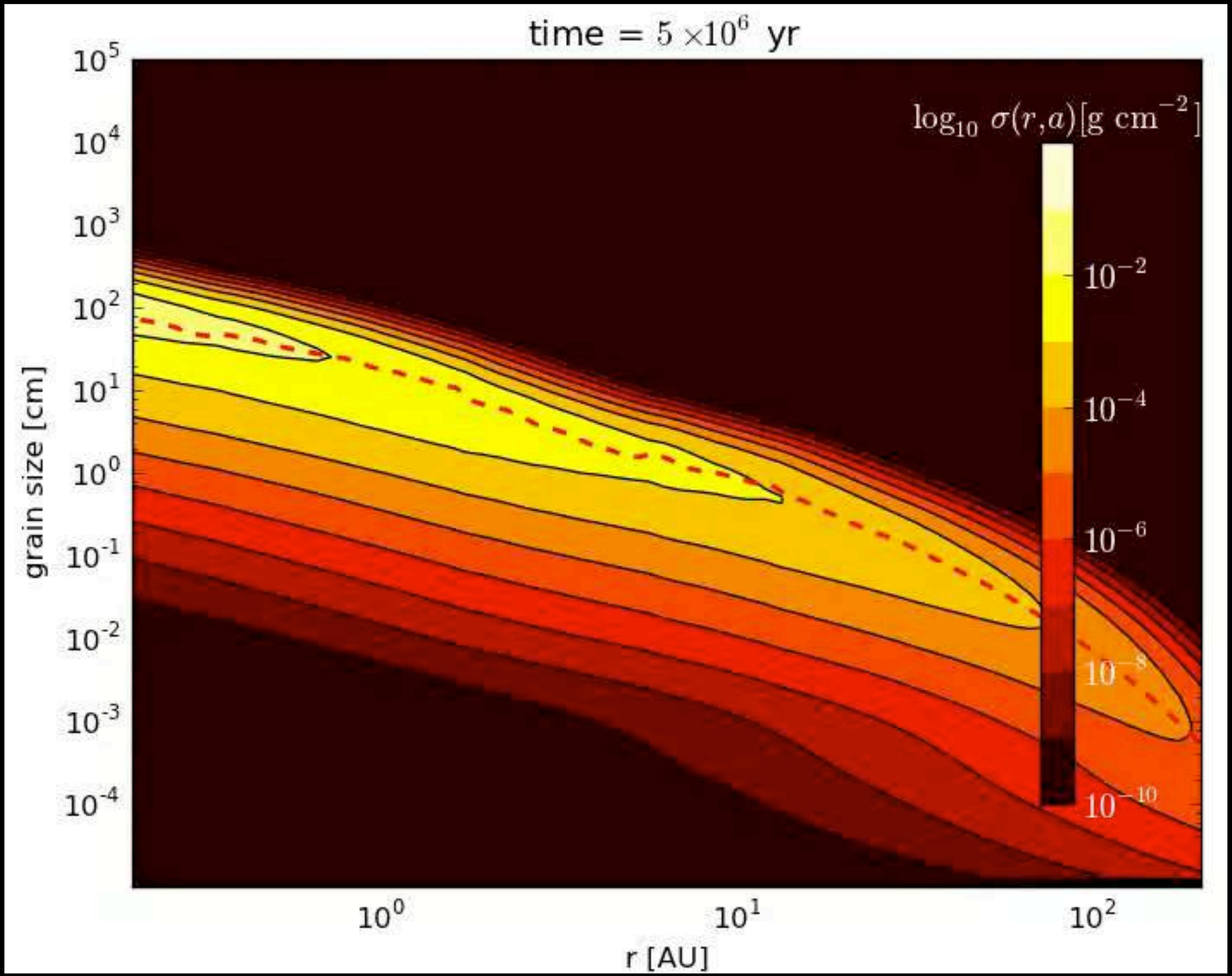
Only grain growth



Birnstiel et al. 2010, 2012

GROWTH BARRIERS

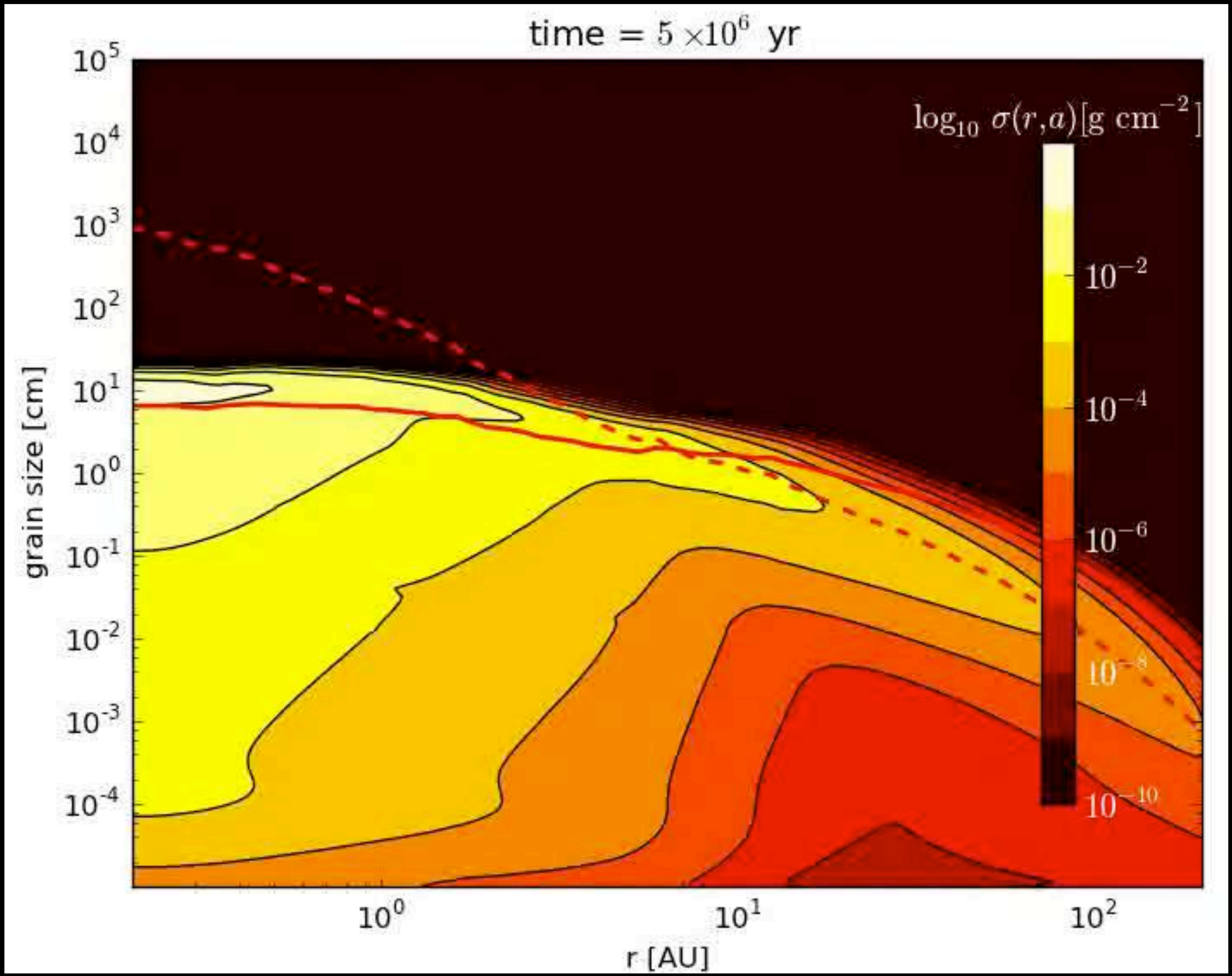
Grain growth and drift



Birnstiel et al. 2010, 2012

GROWTH BARRIERS

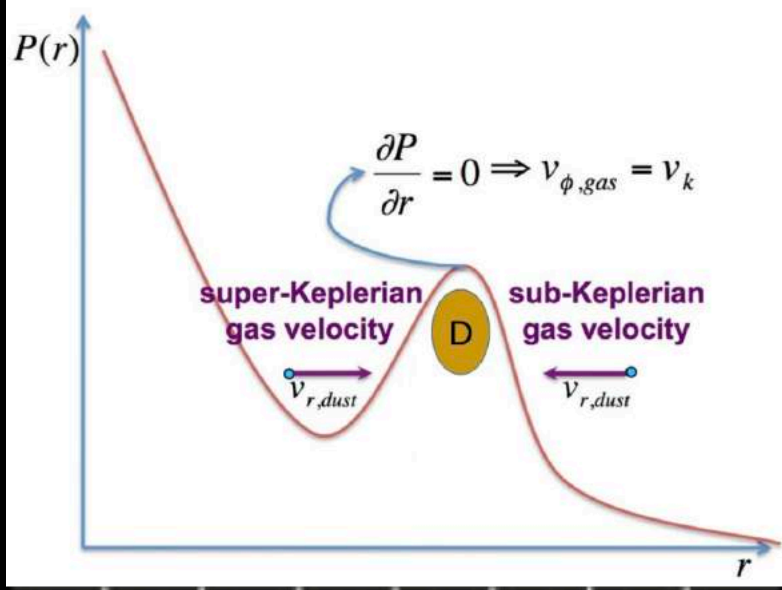
Grain growth, drift, and fragmentation

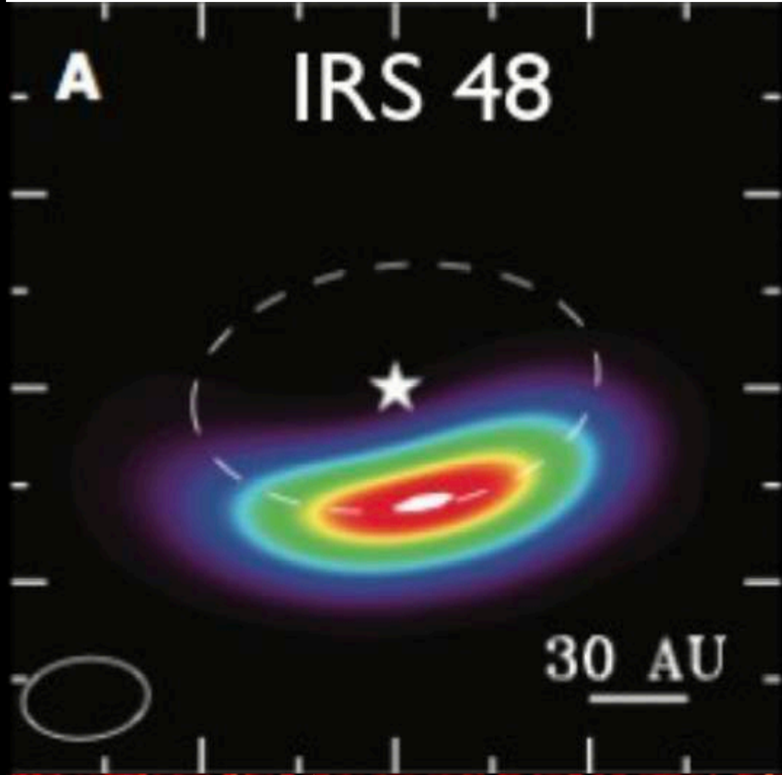


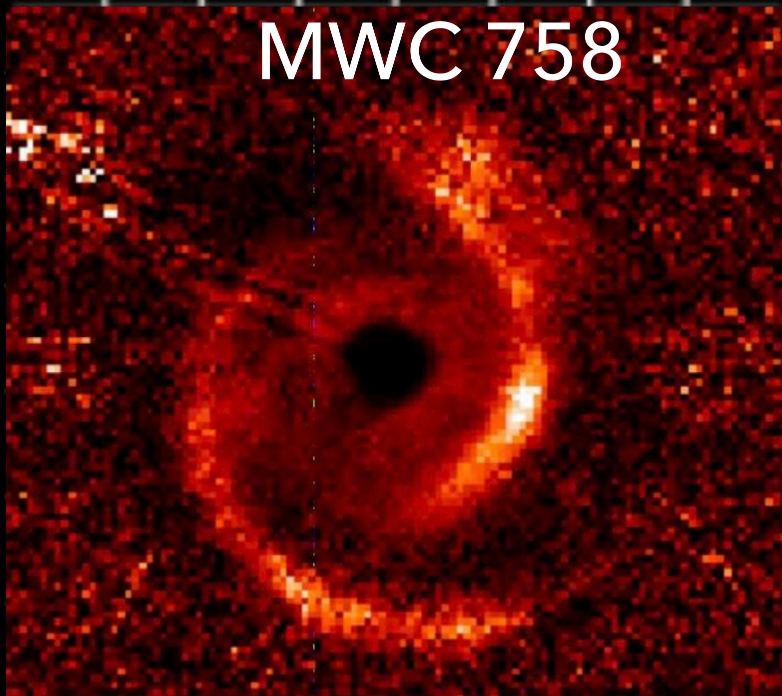
Birnstiel et al. 2010, 2012

OVERCOMING GROWTH BARRIERS

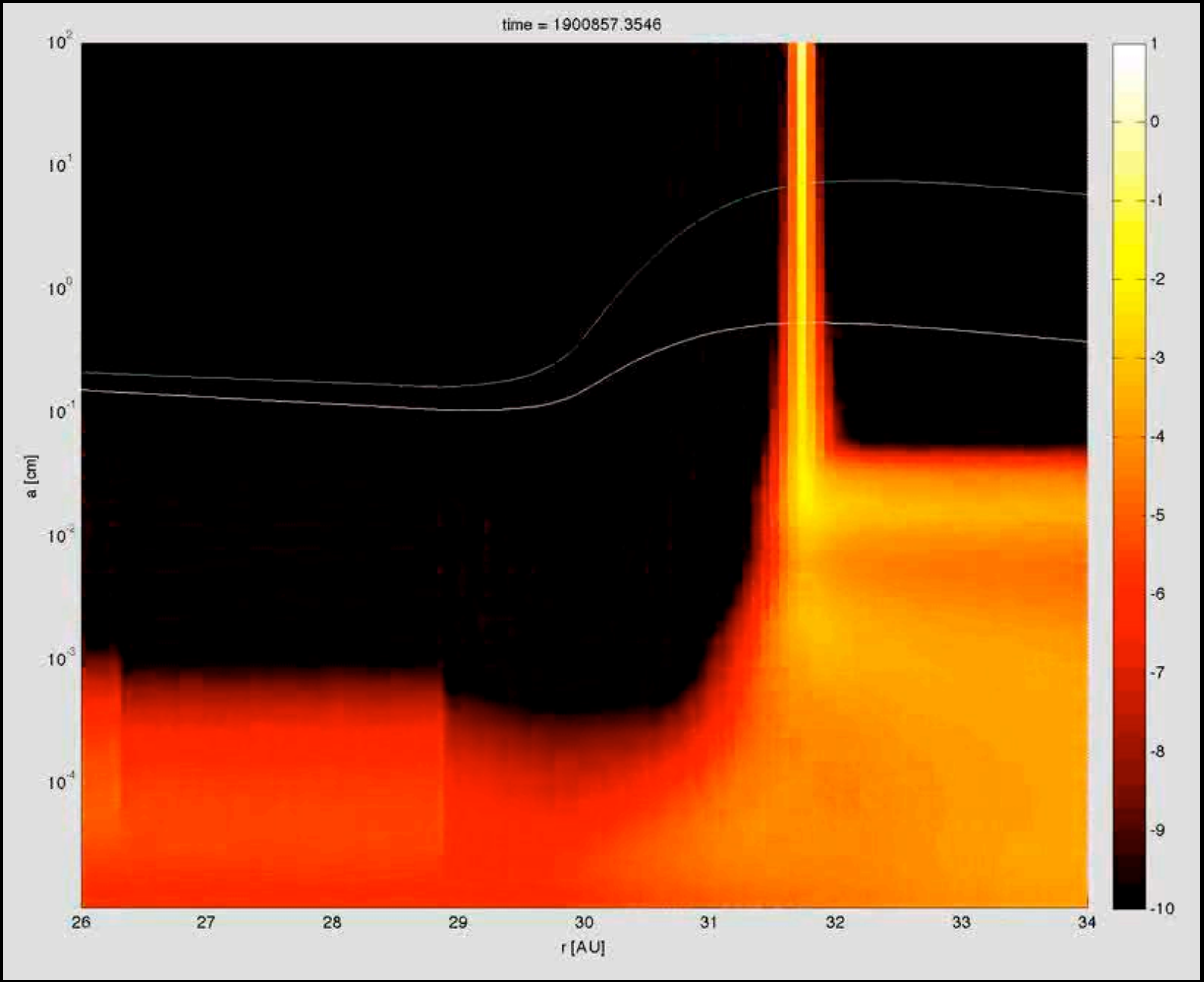
- Dust traps: usually associated with pressure maxima (zero gradient) → no radial or azimuthal drift.
    - Snow lines, turbulence, vortices, planet gaps, gravity, self-induced pile-ups.
- Trap larger grains, small grains follow gas (accretion and viscous spreading). Relative velocities only due to turbulence. Thus for small α, growth can continue.
- A few "lucky" particles in the tails of the velocity distribution may be able to grow to reach planetesimal sizes.



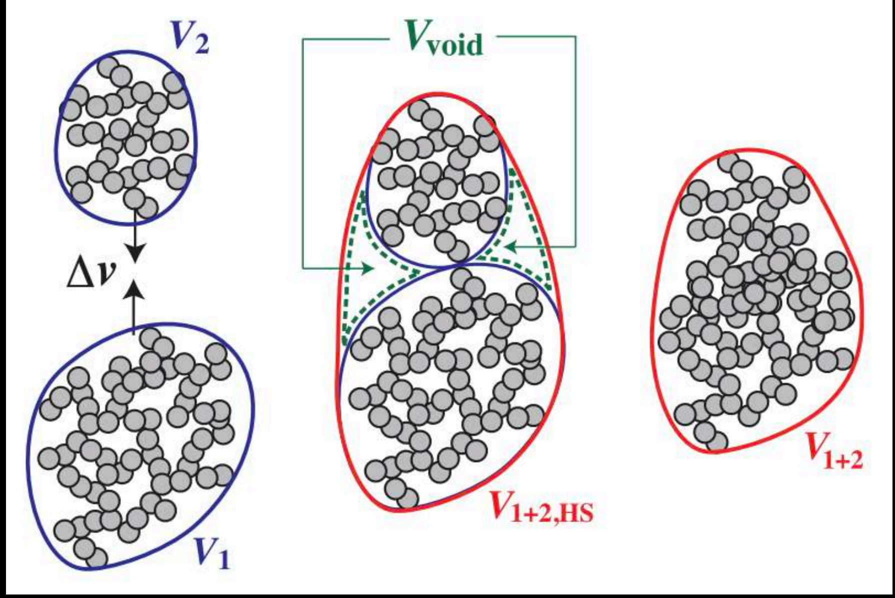




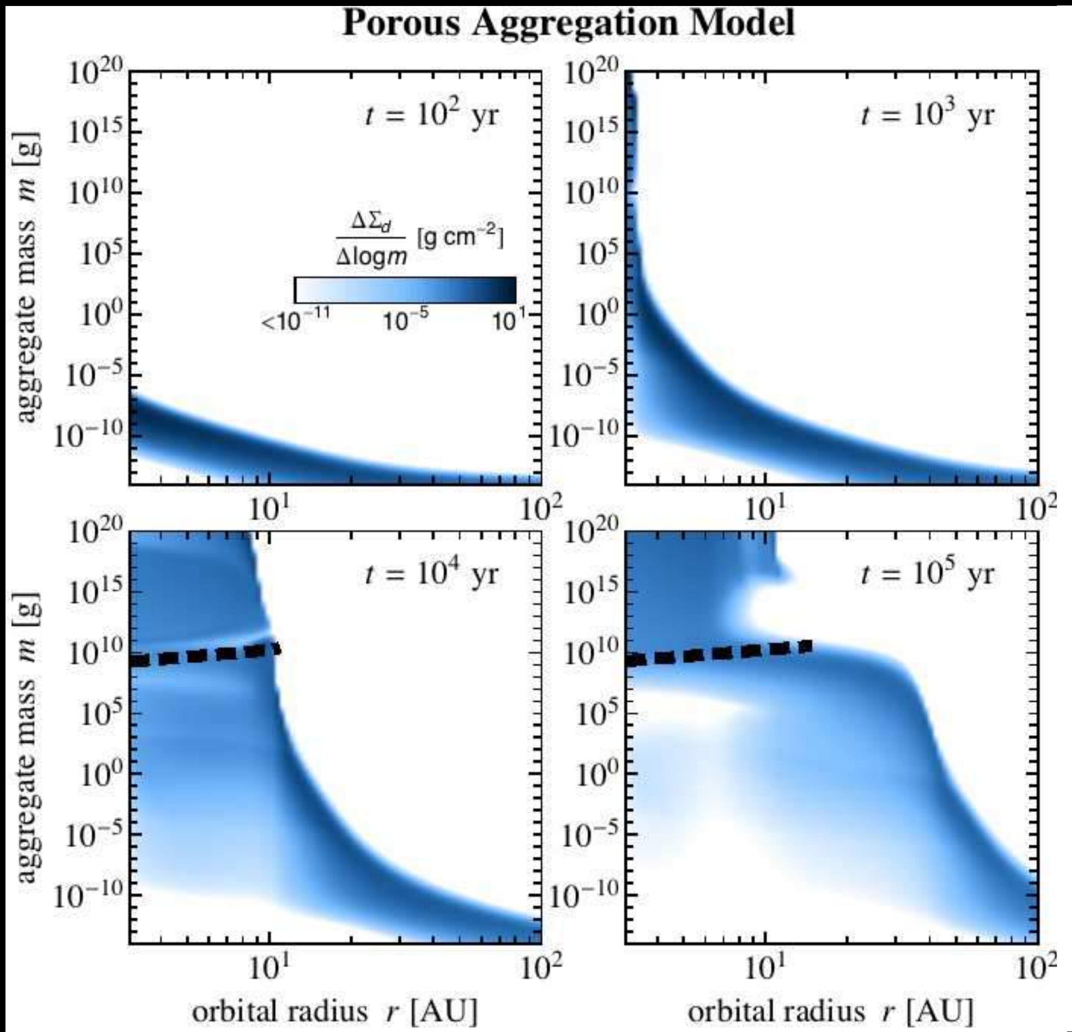
DUST TRAPS: PRESSURE BUMPS

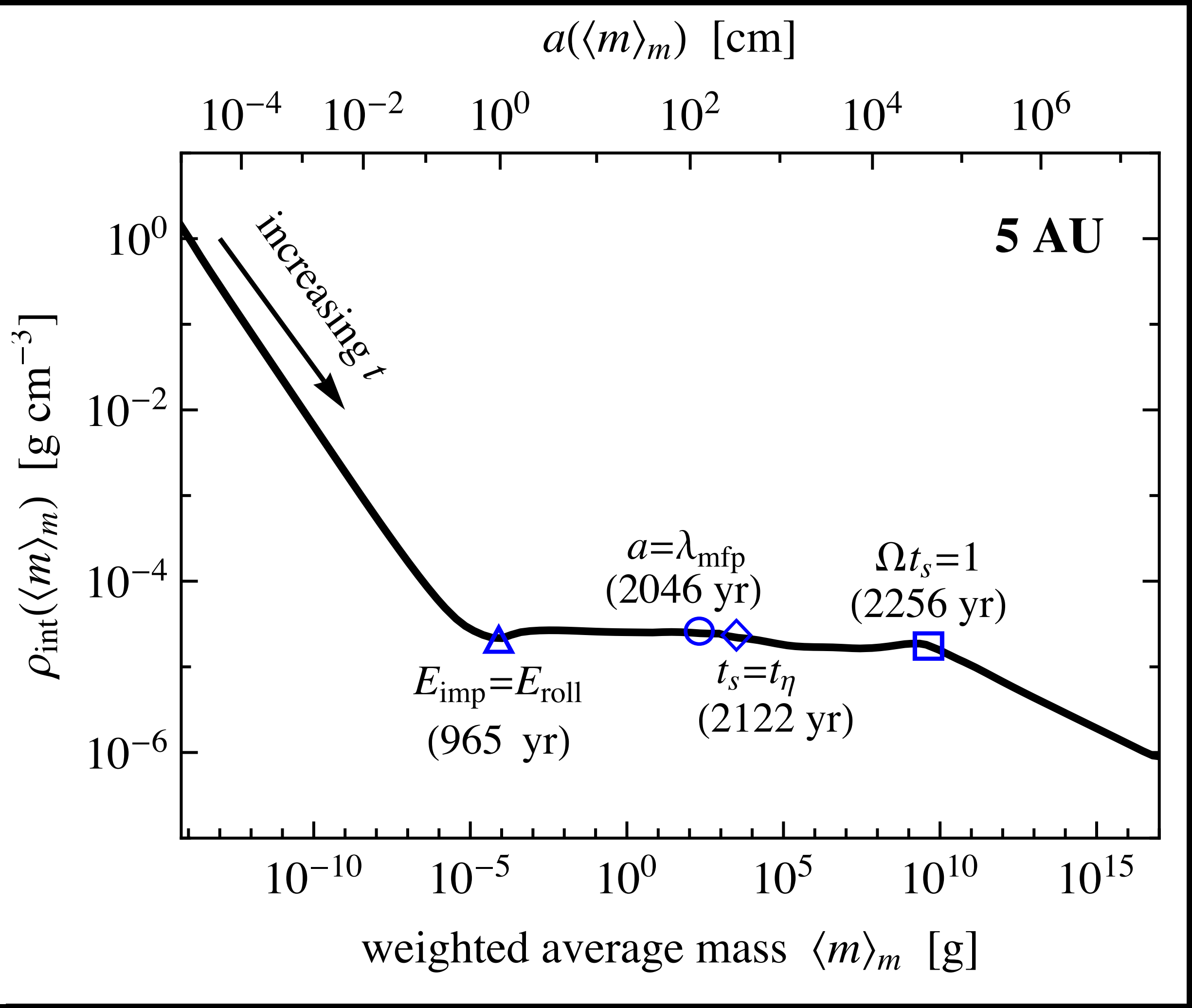


DUST TRAPS: SELF-INDUCED PILE-UPS



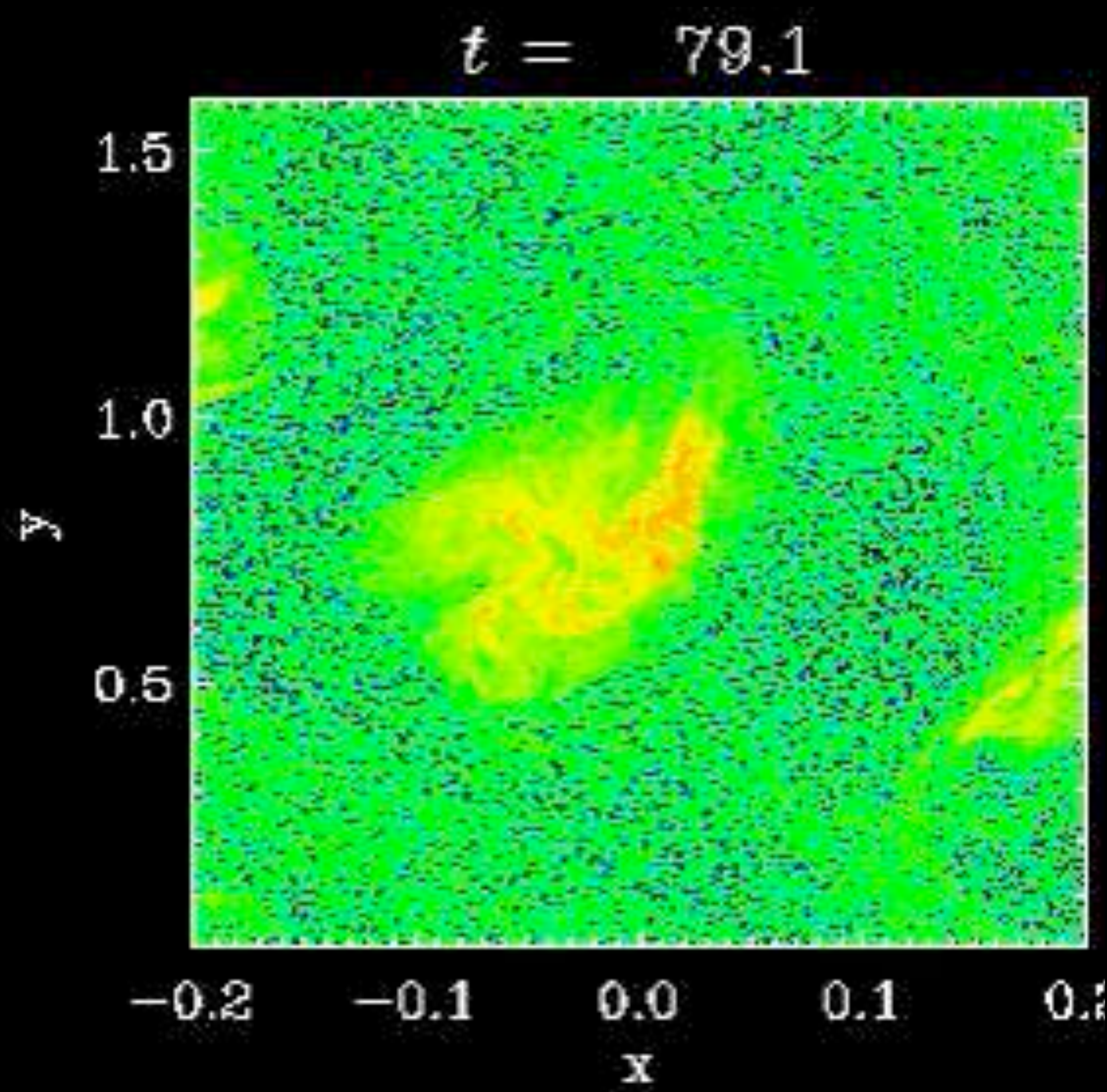
- Fractal particles could potentially break through the drift barrier.
- If $v_{\rm frag} \gtrsim 35$ m/s for icy particles and no significant compaction occurs (e.g. collision energies go into stretching), they break through in the Stokes regime.

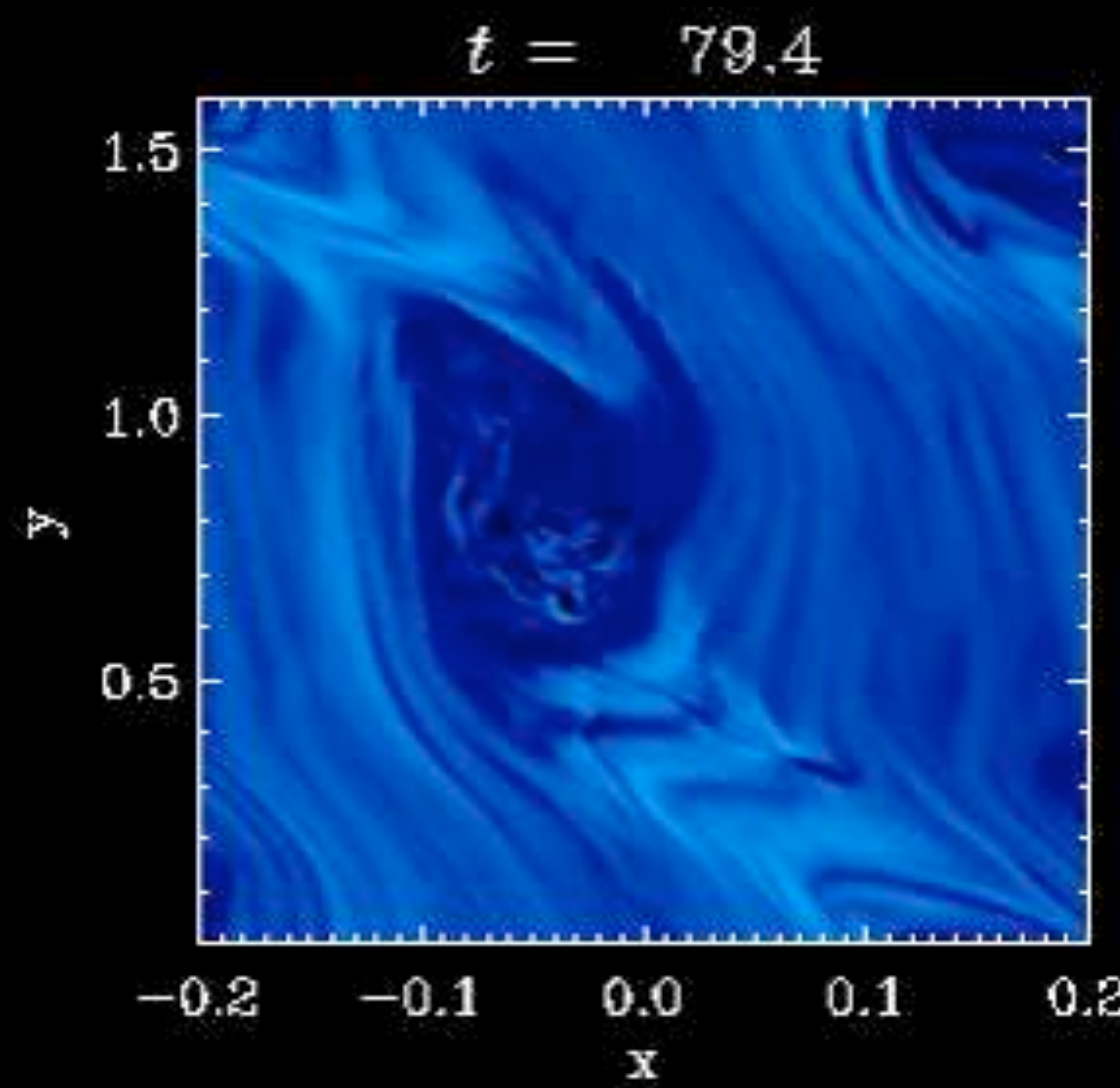


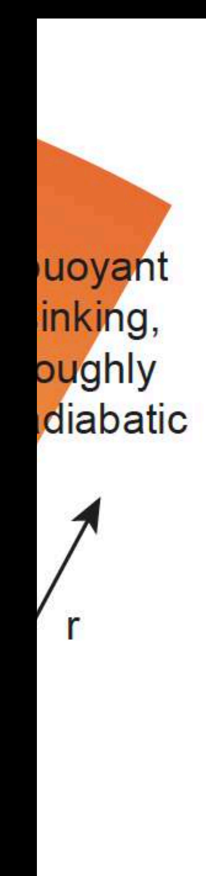


DUST TRAPS: VORTICES

- Vortices can be produced by, e.g., the Rossby-Wave Instability and Baroclinic Instability.
- Anticyclonic vortices are high pressure regions → they capture dust.







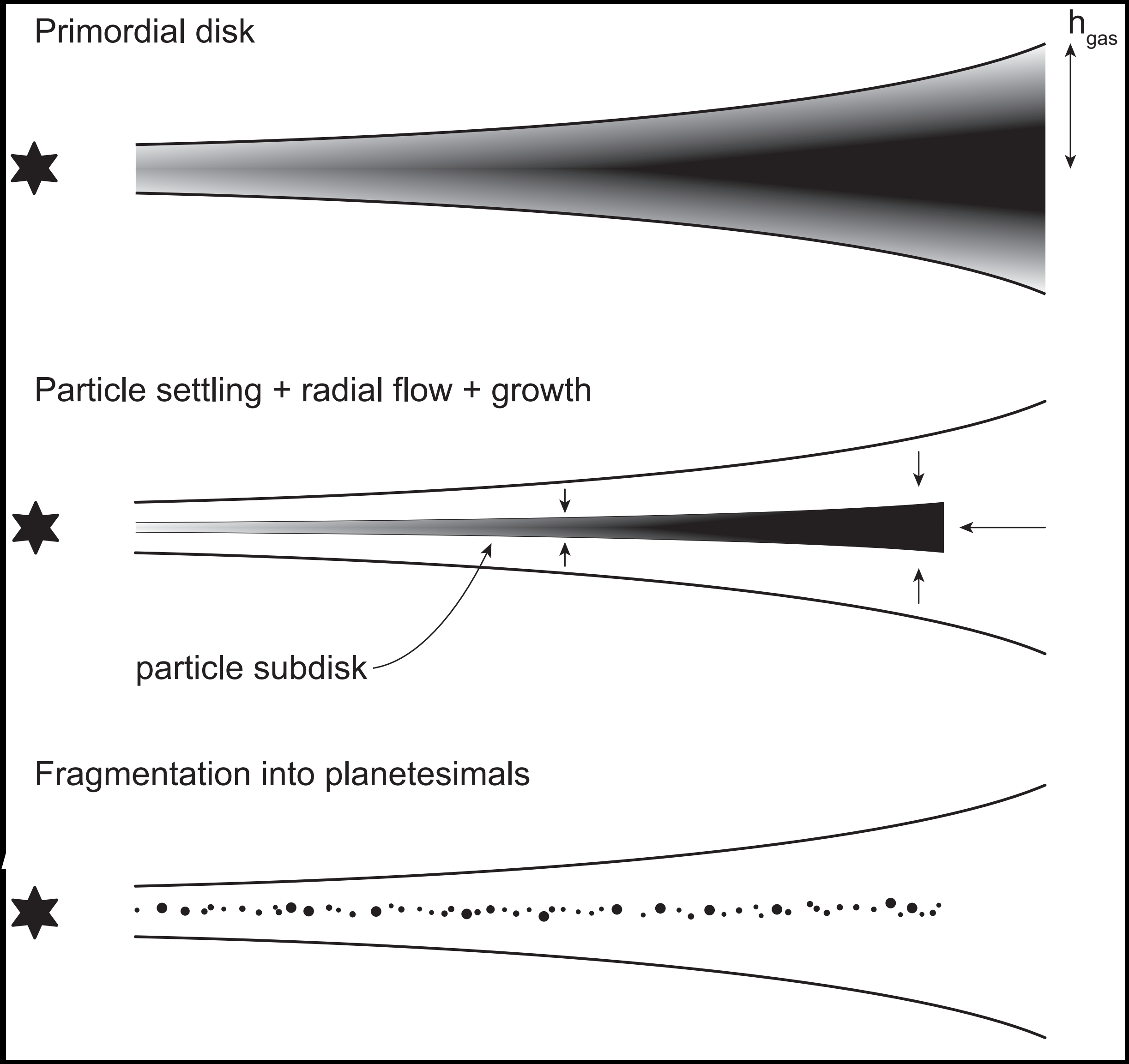
GRAVITATIONAL INSTABILITY (GI)

- **Goldreich-Ward instability**: settling of small grains increases the dust-to-gas ratio at the disc mid-plane, until the dust layer becomes gravitationally unstable and fragments.

- **Toomre criterion**: the disc is unstable for $Q \lesssim 1$, where

$$Q \equiv \frac{c_s \Omega_K}{\pi G \Sigma}$$

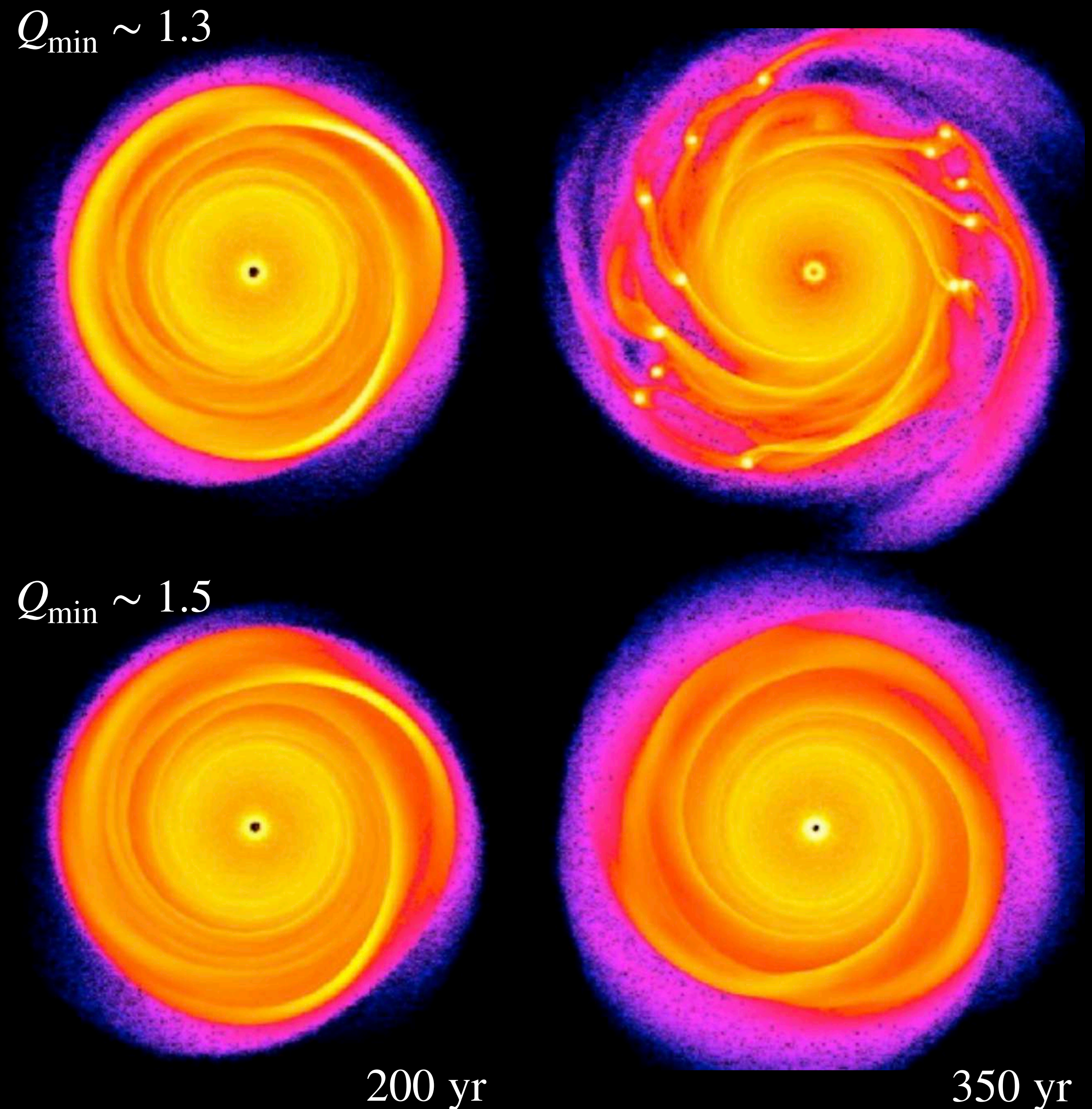
- If $\Sigma_{gas} \sim 100$–$1000\ g/cm^2$ and the dust-to-gas ratio is 0.01, then $Q < 1$ requires a disc temperature less than 1 K!

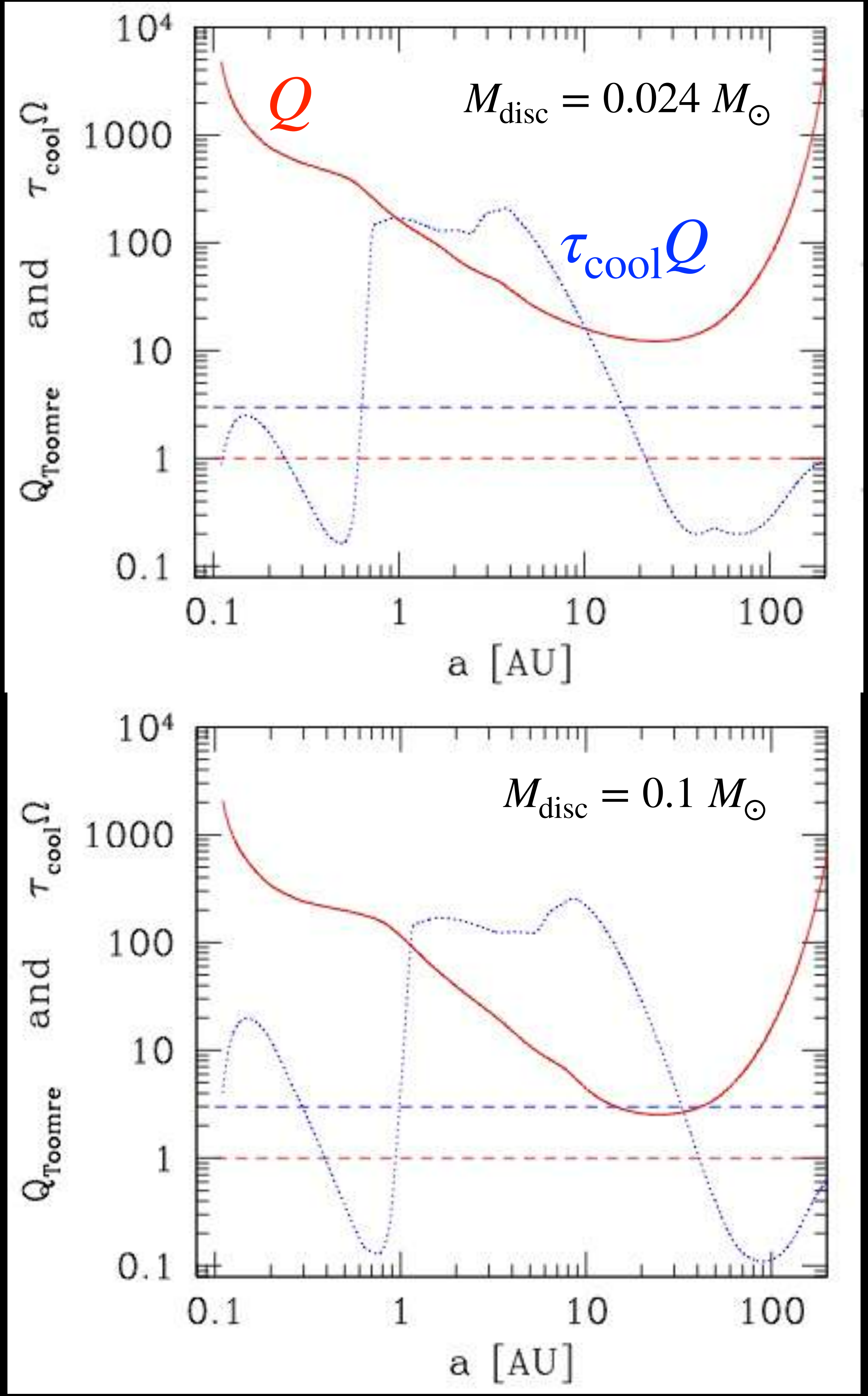


GRAVITATIONAL INSTABILITY (GI)

- The Toomre criterium describes stability against axisymmetric radial rings, but discs become unstable to non-axisymmetric perturbations (spiral waves) at about $Q_{\rm crit} = 1.4$–2.
- The Toomre criterion is necessary, but not sufficient for collapse. Fragmentation into bound clumps requires the cooling timescale to be shorter than the shearing timescale (~orbital period) which acts to disrupt the clump: $\tau_{\rm cool}\Omega_{\rm K} \lesssim \xi$ where ξ is of order unity.
- The spiral waves efficiently transport angular momentum outwards and liberate gravitational binding energy (increases T and reduces Σ, both which reduce Q). The disk reaches a steady state of marginal instability without fragmentation.
- Explains why we don't see disc masses comparable to the star.

GRAVITATIONAL INSTABILITY (GI)



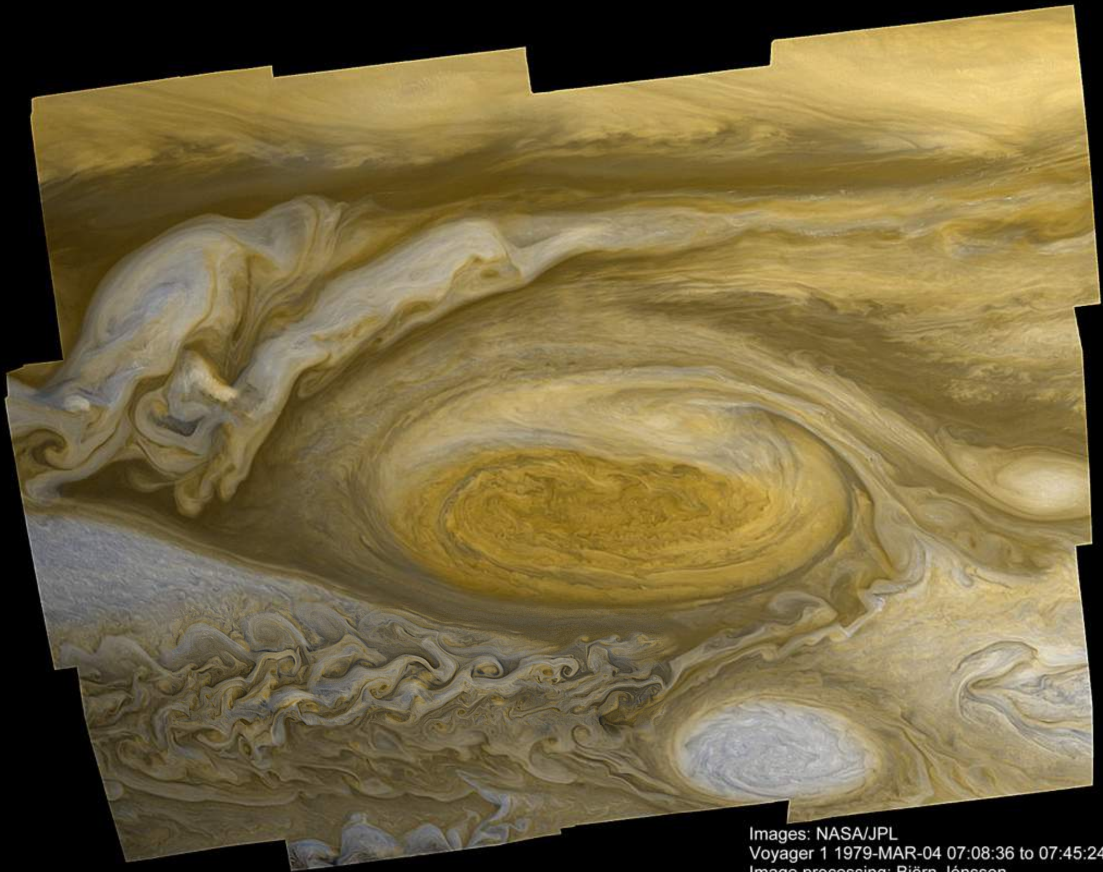


STREAMING INSTABILITY (SI)

- Dust experiences a headwind in discs, but if the dust layer of large grains (pebbles) is sufficiently compact and dense ($\sim 10^4 \times$ thinner and $\sim 100 \times$ denser than the gas!) then the dust accelerates the gas and reduces the headwind it feels. This has two consequences:
    - Radial drift is halted and dust drifting in from outside piles up.
    - The accelerated gas causes a pressure bump (dust trap).
- The process rapidly runs away until the clump becomes self-gravitating and collapses to form planetesimals.

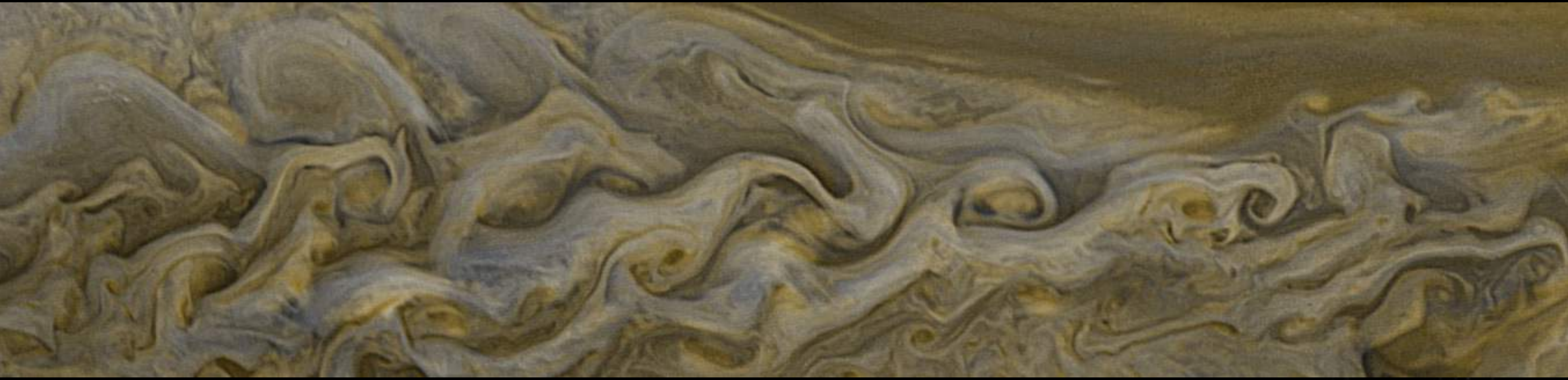
STREAMING INSTABILITY (SI)

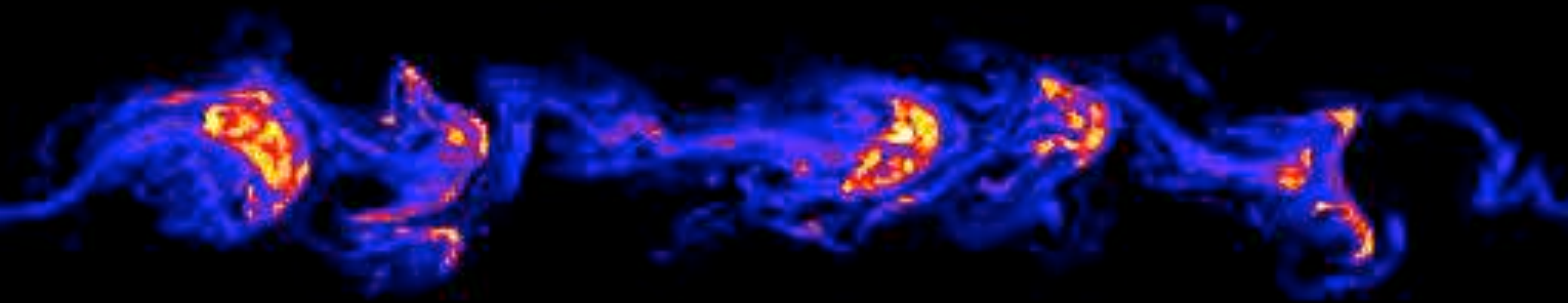
- While the compact dust layer is dynamically dominated by the dust, the layers above are still dominated by the gas → large vertical shear.
- Kelvin-Helmholtz Instability develop which increases the velocity dispersion of the dust layer.



Images: NASA/JPL
Voyager 1 1979-MAR-04 07:08:36 to 07:45:24
Image processing: Björn Jónsson

STREAMING INSTABILITY (SI)





FROM UNIVERSE

TO PLANETS

LECTURE 3.3: PLANETESIMALS

PLANETESIMAL FORMATION

354P/LINEAR

PLANETESIMAL FORMATION

- First collision of main astroid belt object detected on 6 January 2010.

- Its orbit in the main astroid belt, the never-before-seen X pattern (which remained intact), and the nucleus outside the main halo rule out the possibility of a comet.

- Probably created by the impact of a small m-size object on the larger asteroid (~150 m) in February/March 2009.

- Particle sizes in the tail are probably between 1 mm and 2.5 cm in diameter. The tail contains enough dust to make a sphere of diameter 20 m.

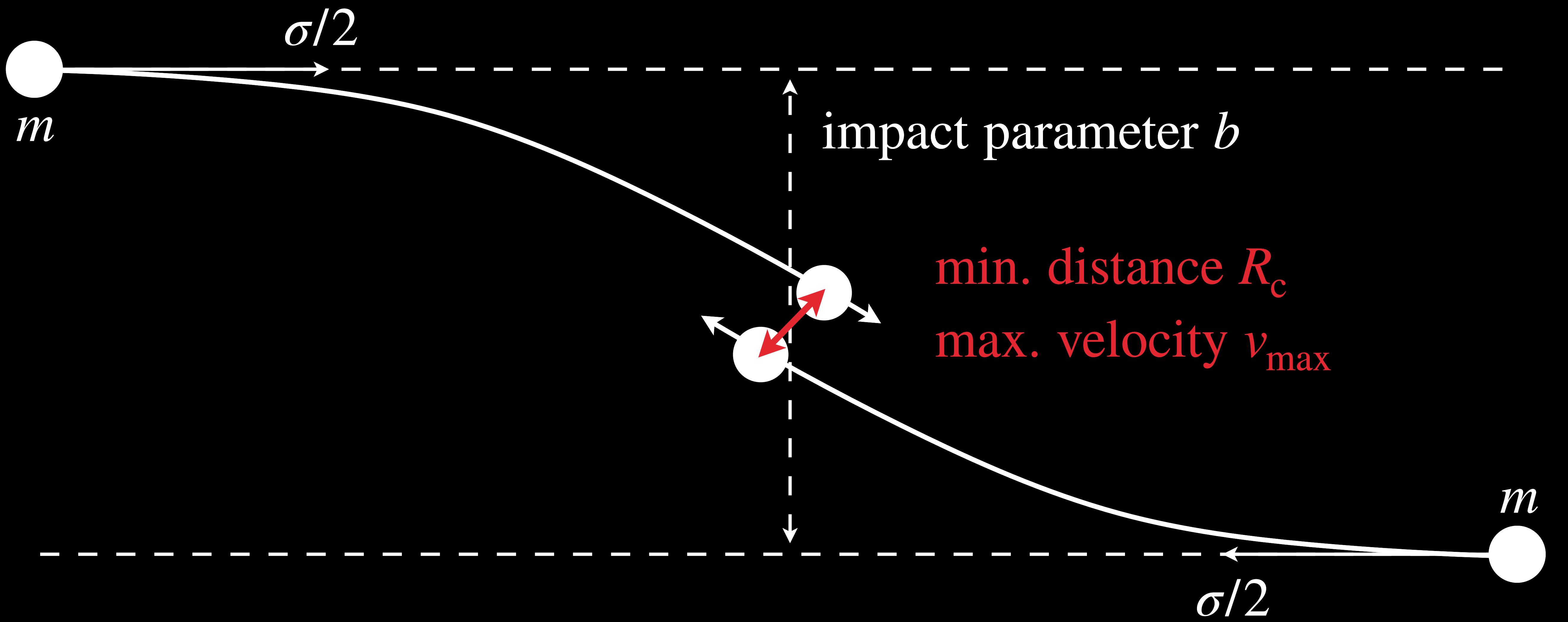
PLANETESIMAL FORMATION

via Coagulation	via Gravity
PROS	**PROS**
‣ Dust growth surely happens. ‣ Effects confirmed in the lab. ‣ Various mechanisms (ices, organics, velocity distribution) suggest the barriers have holes.	‣ Well studied process, shown to work numerically. ‣ Time scales are shorter than drift time scale. ‣ Some observational evidence for collections of small pebbles.
CONS	**CONS**
‣ Collision velocities increase → no more sticking (?). Hard to experiment with boulders. ‣ Formation time scales often too long compared to drift time scales.	‣ Turbulence in disks not well understood. ‣ Needs high dust-to-gas ratios. ‣ Needs large numbers of pebbles (1mm–100 cm).

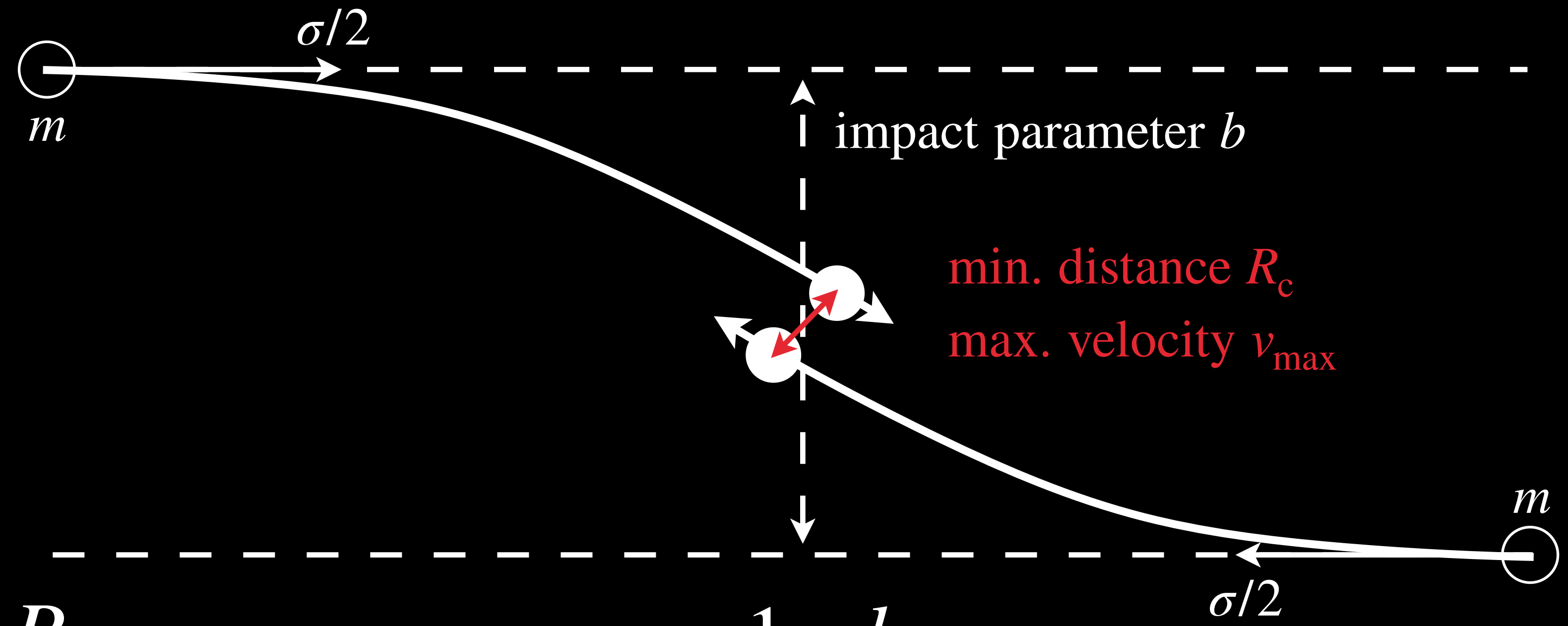
PLANETESIMAL OVERVIEW

- Problems we face in understanding planetesimal dynamics:
  - Number of 5 km bodies to get the total mass of terrestrial planets is $\sim 4 \times 10^9$.
  - They interact/collide over Myr–Gyr timescales.
- What is needed for a complete model:
  - Understand how eccentricity e, inclination i, and mass m evolve with time t.
  - Derive a collision rate for the planetesimal distribution and a statistical treatment for smaller bodies: $f(m, e, i)$.
  - Predict the outcome of a collision given m_1, m_2, and Δv.

GRAVITATIONAL FOCUSING



GRAVITATIONAL FOCUSING



- Angular momentum conservations gives:

$$J = 2 \cdot m\frac{\sigma}{2} \cdot \frac{b}{2} = 2 \cdot m v_{\max}\frac{R_c}{2} \quad \longrightarrow \quad v_{\max} = \frac{1}{2}\frac{\sigma b}{R_c}$$

- Conservation of energy gives (upon inserting $v_{\max}$):

$$E = 2 \cdot \frac{1}{2}m\left(\frac{\sigma}{2}\right)^2 = 2 \cdot \frac{1}{2}m v_{\max}^2 - \frac{Gm^2}{R_c} \quad \longrightarrow \quad b^2 = R_c^2 + \frac{4GmR_c}{\sigma^2}$$

- Collisions only occur if $R_c < R_s$, where R_s is the sum of the sizes. Using the escape velocity ($v_{esc}^2 = 4Gm/R_s$):

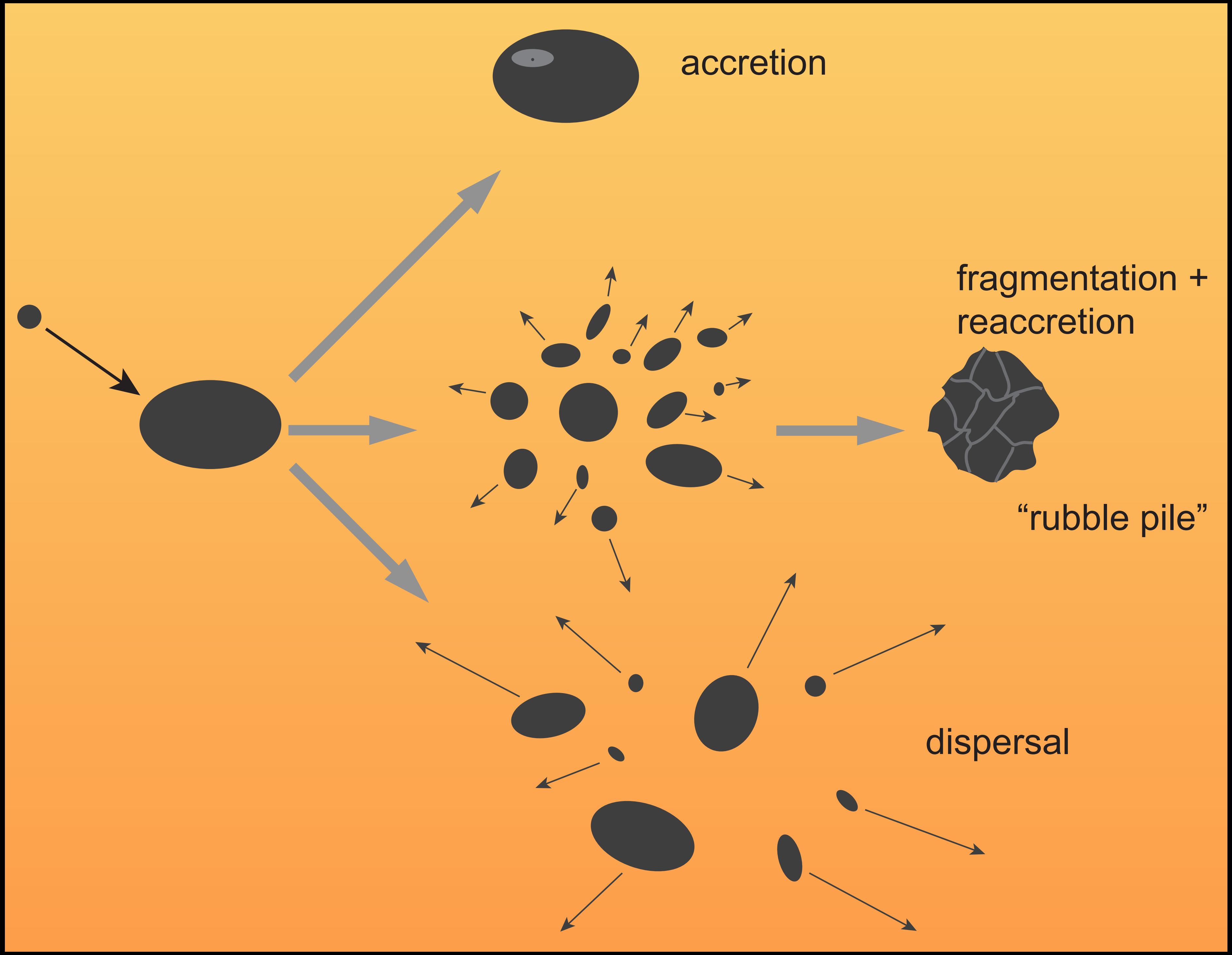
maximum distance leading to a collision

$$b^2 = R_s^2\left(1 + \frac{v_{esc}^2}{\sigma^2}\right)$$

collision cross-section (also valid for different m)

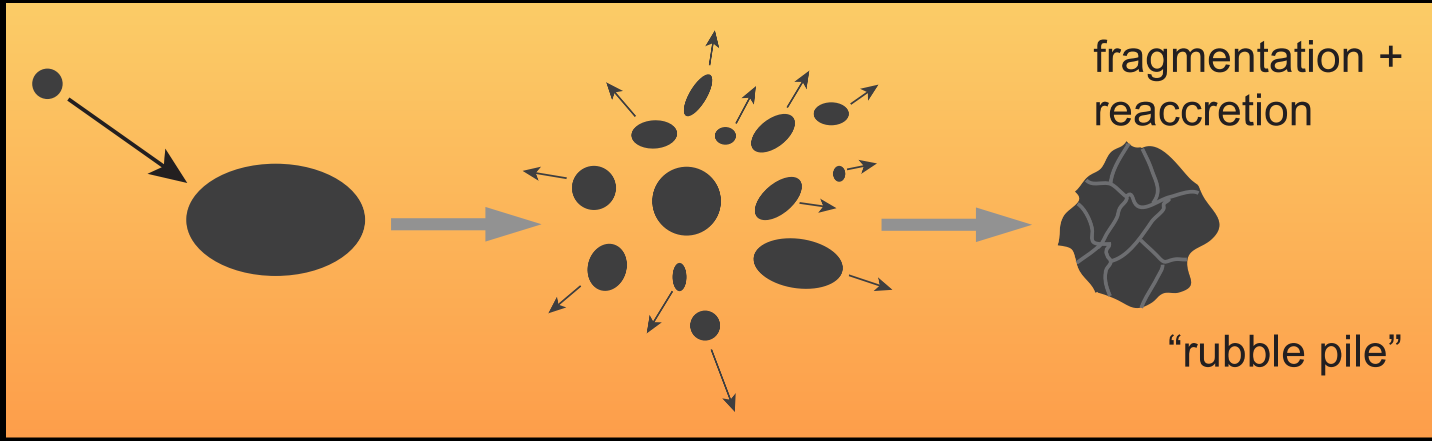
$$\Gamma = \underbrace{\pi R_s^2}_{\Gamma_{geo}}\left(1 + \frac{v_{esc}^2}{\sigma^2}\right)$$

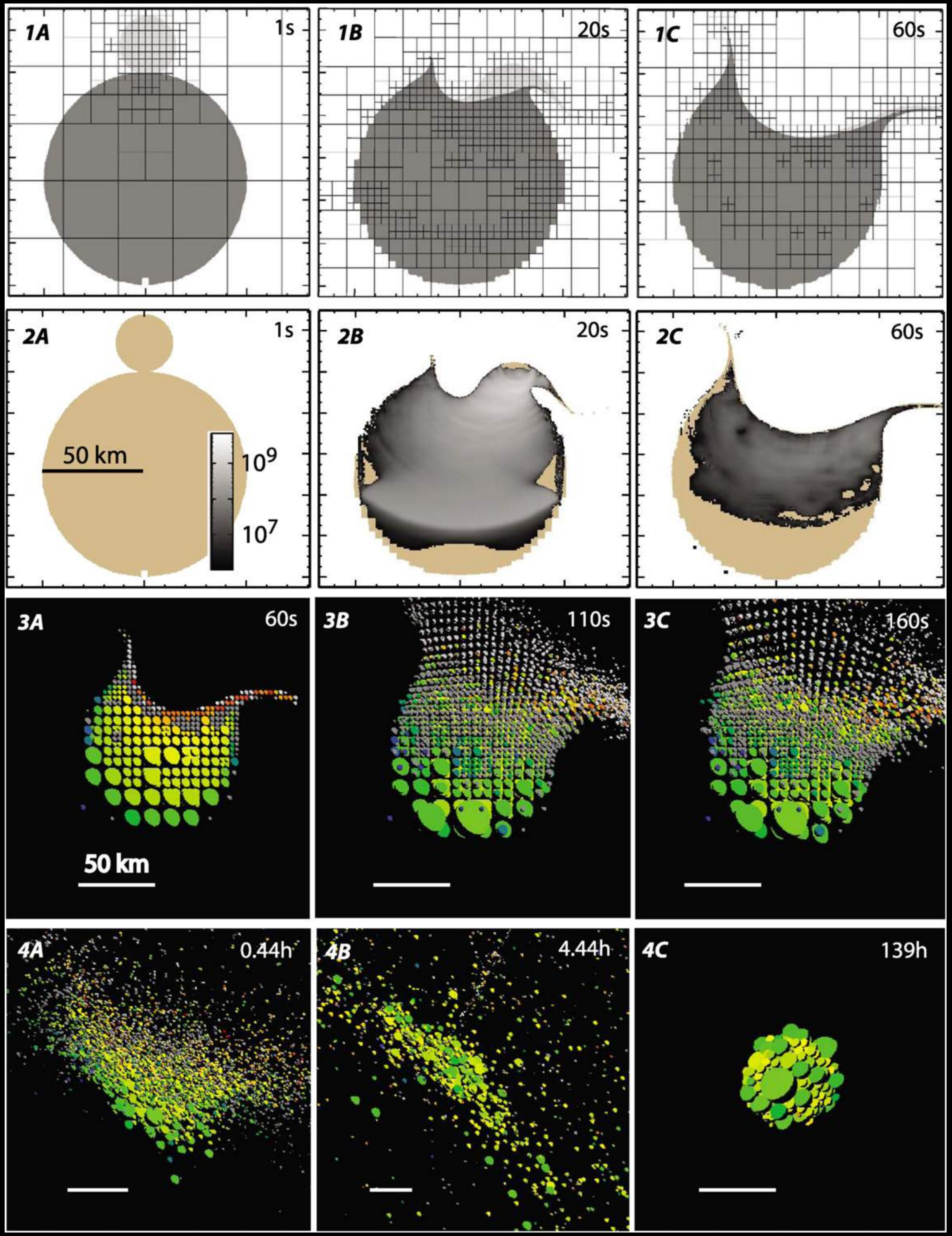
PLANETESIMAL COLLISIONS



PLANETESIMAL COLLISIONS

- Example of a 3D, 45 degree, impact between two basalt spheres ($R_{\text{proj}} = 14$ km, $R_{\text{tar}} = 50$ km, $v_{\text{i}} = 1.8$ km s^{-1}).
- Row 1: projectile (light grey), target (dark grey), and adaptive mesh.
- Row 2: shows the projectile and target (beige) and the pressure due to the impact (grey scale in Pa).
- Rows 3–4: Colors represent the peak pressure attained during the impact (logarithmic range of 0.01 to 7 GPa). The last frame shows only the largest reaccumulated, post-collision remnant which equilibrates to 45% of the target mass in this simulation.





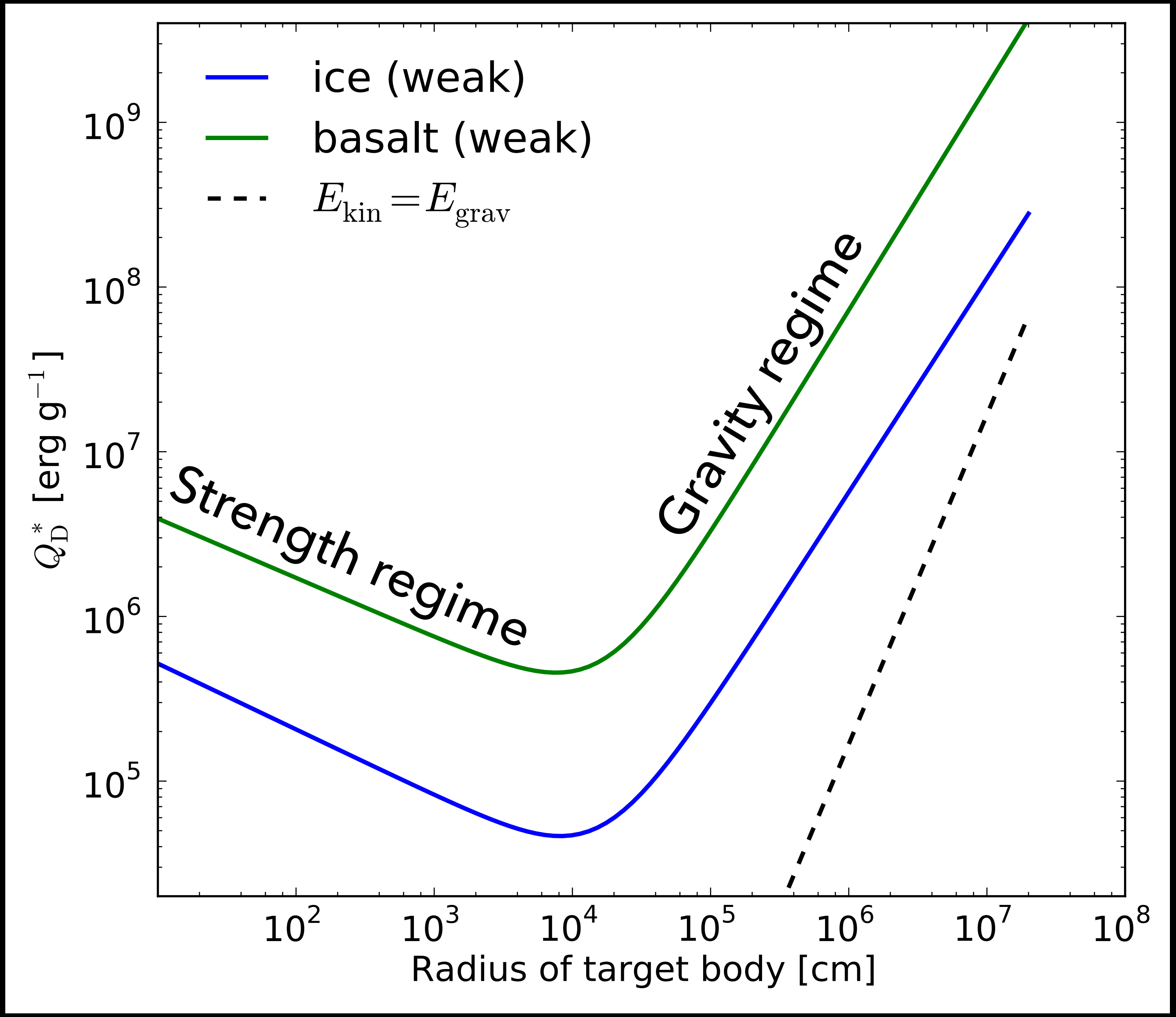
GRAVITATIONAL BINDING ENERGY

- Specific energy of the impact: $$Q \equiv \frac{mv^2}{2M} = \frac{\text{impactor energy}}{\text{target mass}}$$

- The gravitational binding energy for a sphere of uniform density:

$$E_{\text{grav}} = \frac{3}{5}\frac{GM^2}{R}$$

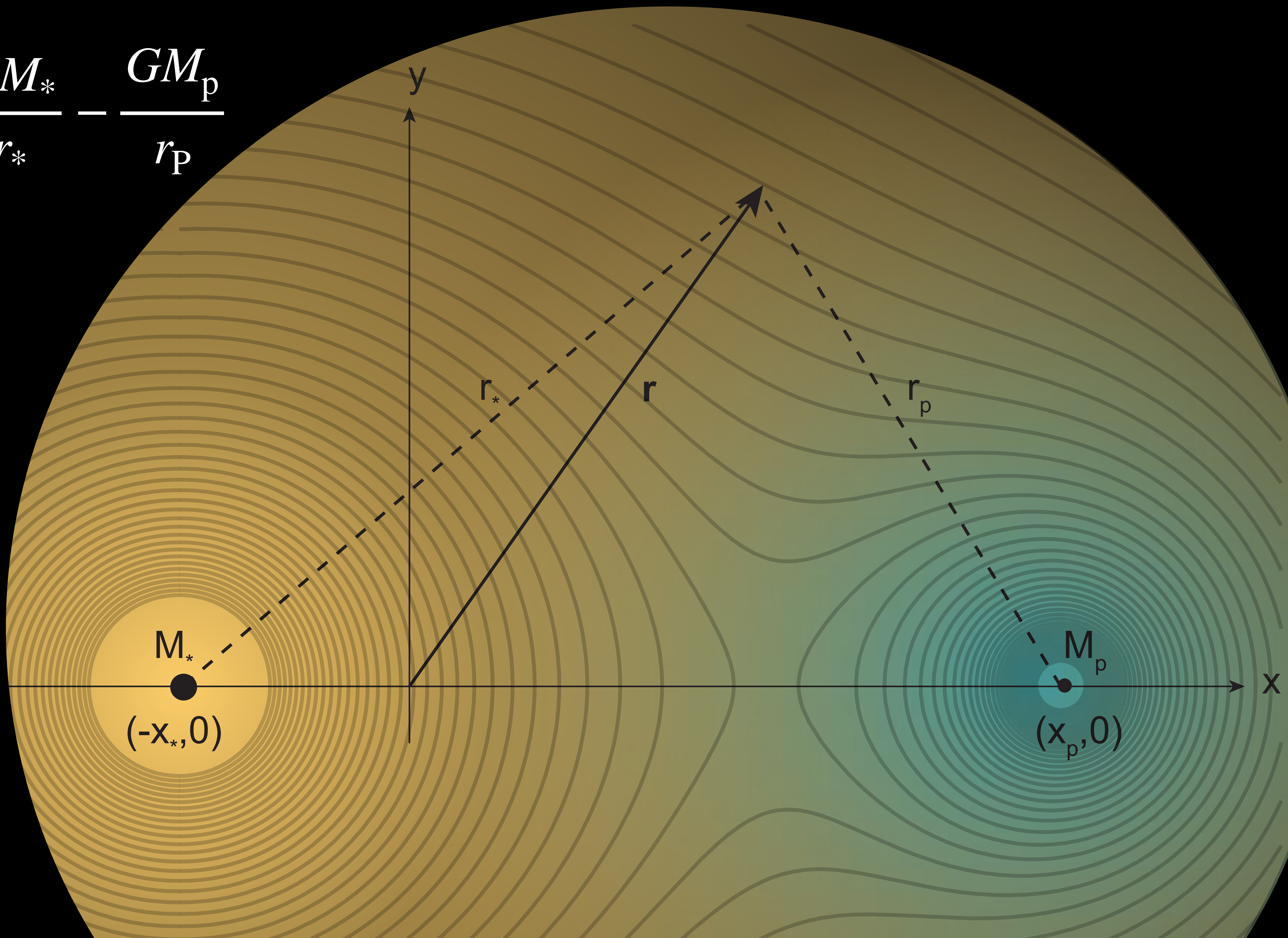
- Energy goes into heating phase changes, ejecta, ect..



HILL RADIUS

$$\ddot{\mathbf{r}} = -\nabla\Phi \overbrace{-2(\boldsymbol{\Omega}_K \times \dot{\mathbf{r}})}^{\text{Coriolis Force}} \overbrace{-\boldsymbol{\Omega}_K \times (\boldsymbol{\Omega}_K \times \mathbf{r})}^{\text{Centrifugal Force}}$$

$$\Phi = -\frac{GM_*}{r_*} - \frac{GM_p}{r_P}$$



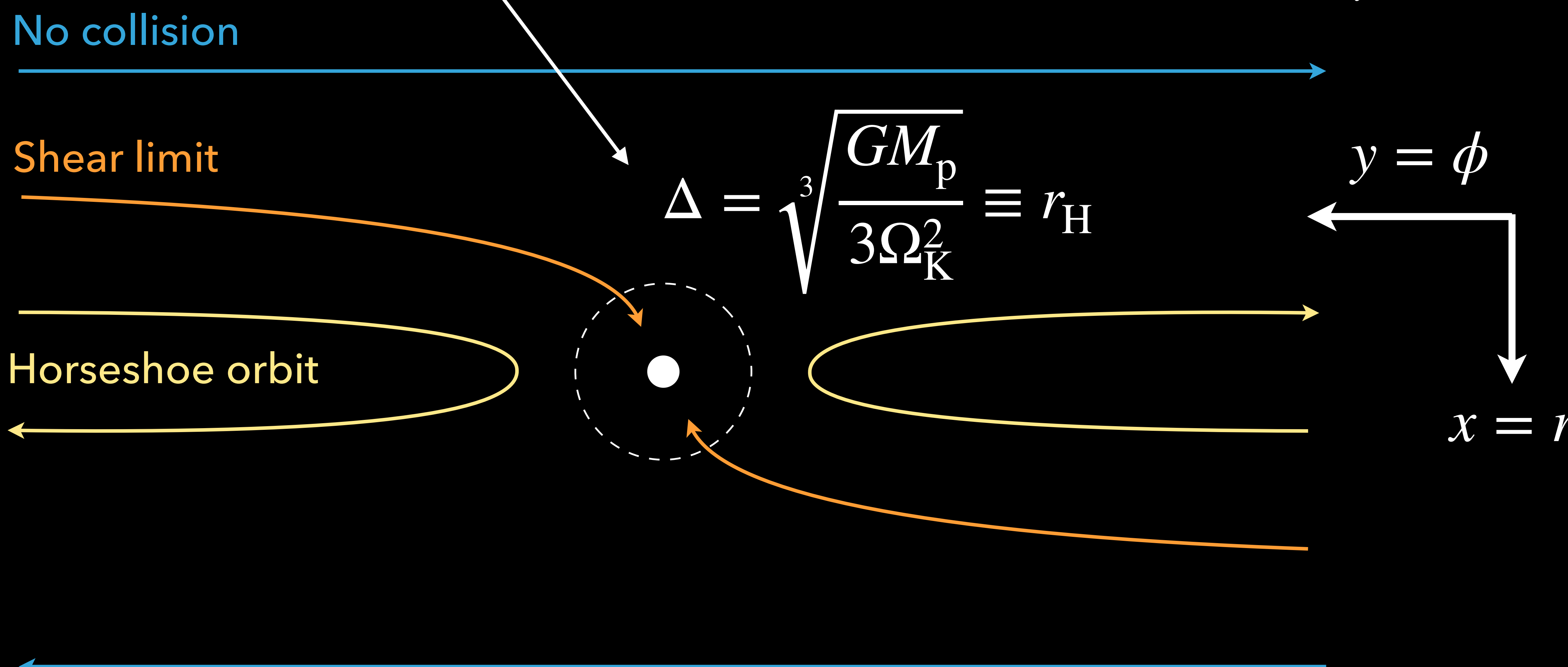
HILL RADIUS

- Assuming $M_* \gg M_\mathrm{P}$ and $\Delta = |\mathbf{r} - \mathbf{r}_\mathrm{P}|$, we can simplify:

$$\ddot{x} - 2\Omega_\mathrm{K}\dot{y} = \underbrace{\left(3\Omega_\mathrm{K}^2 - \frac{GM_\mathrm{p}}{\Delta^3}\right)}_{=0} x \qquad \ddot{y} + 2\Omega_\mathrm{K}\dot{x} = -\frac{GM_\mathrm{p}}{\Delta^3} y$$

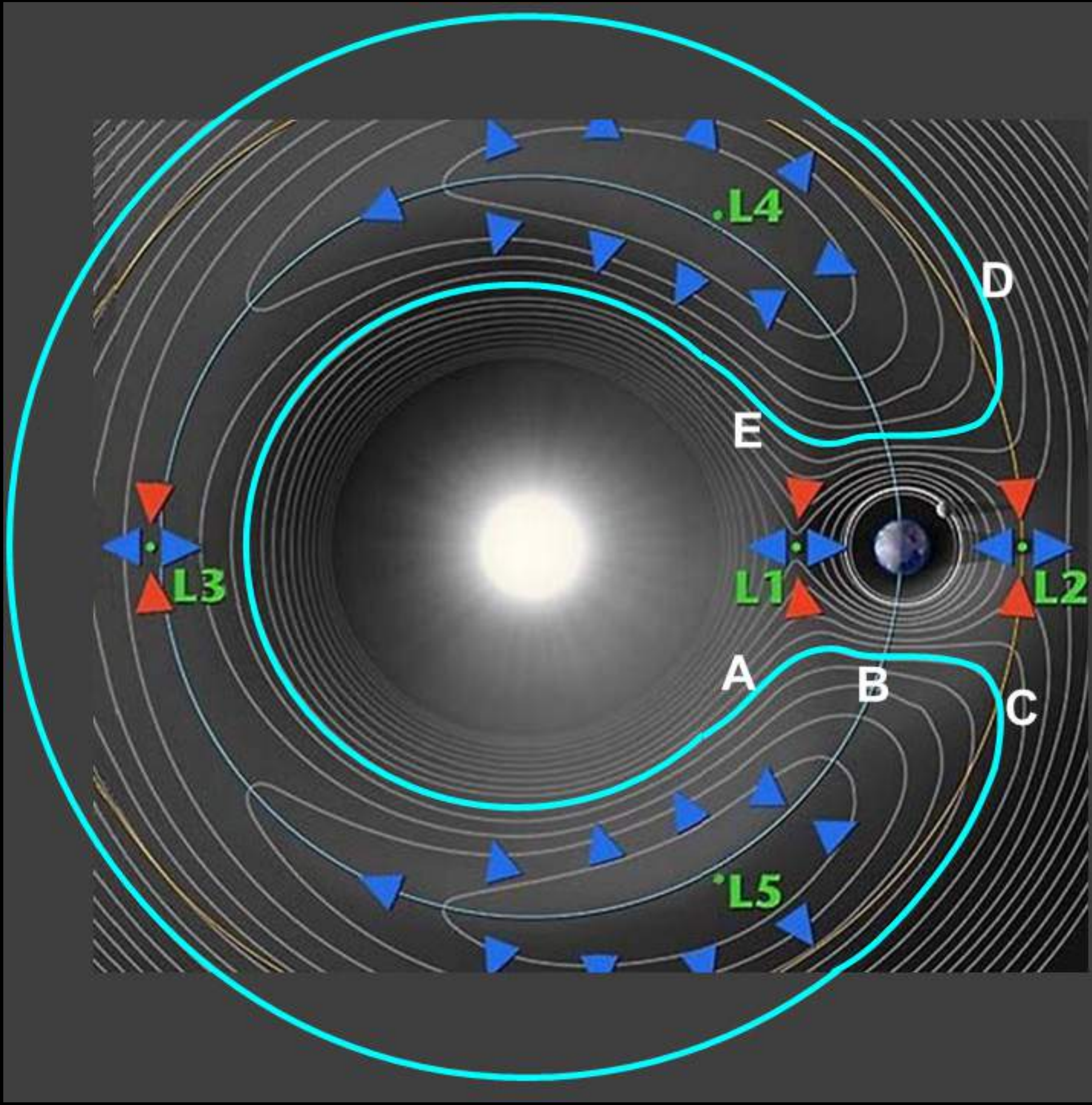
look for where the radial force vanishes (at $y = 0$)

$$\Delta = \sqrt[3]{\frac{GM_\mathrm{p}}{3\Omega_\mathrm{K}^2}} \equiv r_\mathrm{H}$$



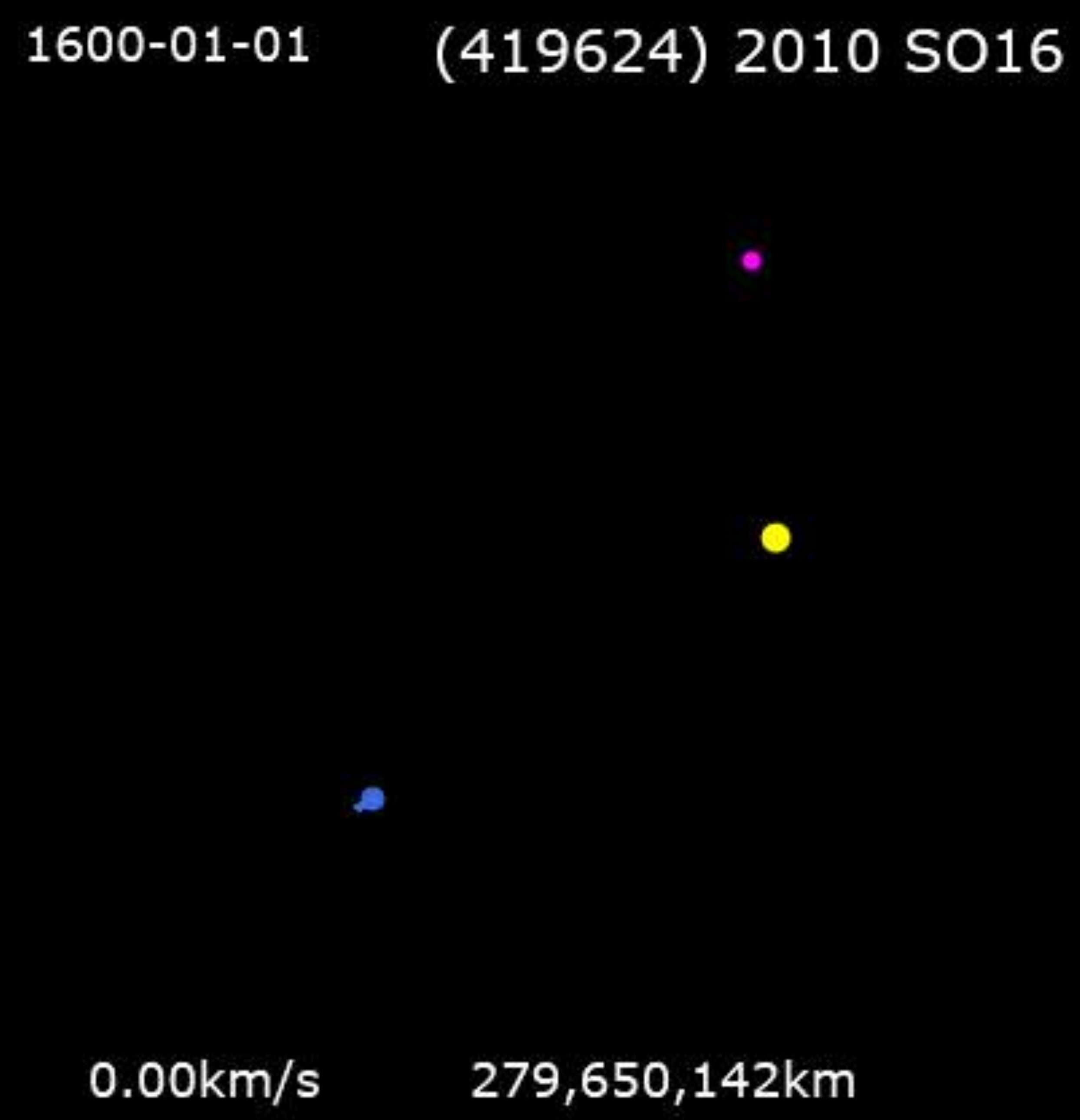
LAGRANGE POINTS

- Lagrange points are locations where the gravitational forces from two larger bodies and the orbital motion of a third body interact to create a stable or semi-stable location.
- Only L4 and L5 are stable (Trojan astroids).



HORSESHOE ORBITS

- Horseshoe orbits is a type of co-orbital motion of a small orbiting body relative to a larger orbiting body.



- The orbital period of the smaller body is very nearly the same as for the larger body, and its path appears to have a horseshoe shape as viewed from the larger object in a rotating reference frame.

CLOSE ENCOUNTERS

- We can define a characteristic velocity (the Hill velocity) as the orbital velocity around the planetesimal at the distance of the Hill radius:

$$v_{\mathrm{H}} = \Omega_{\mathrm{P}} r_{\mathrm{H}} \qquad \text{where} \qquad \Omega_{\mathrm{P}} = \sqrt{\frac{GM_{\mathrm{P}}}{r_{\mathrm{P}}^3}}$$

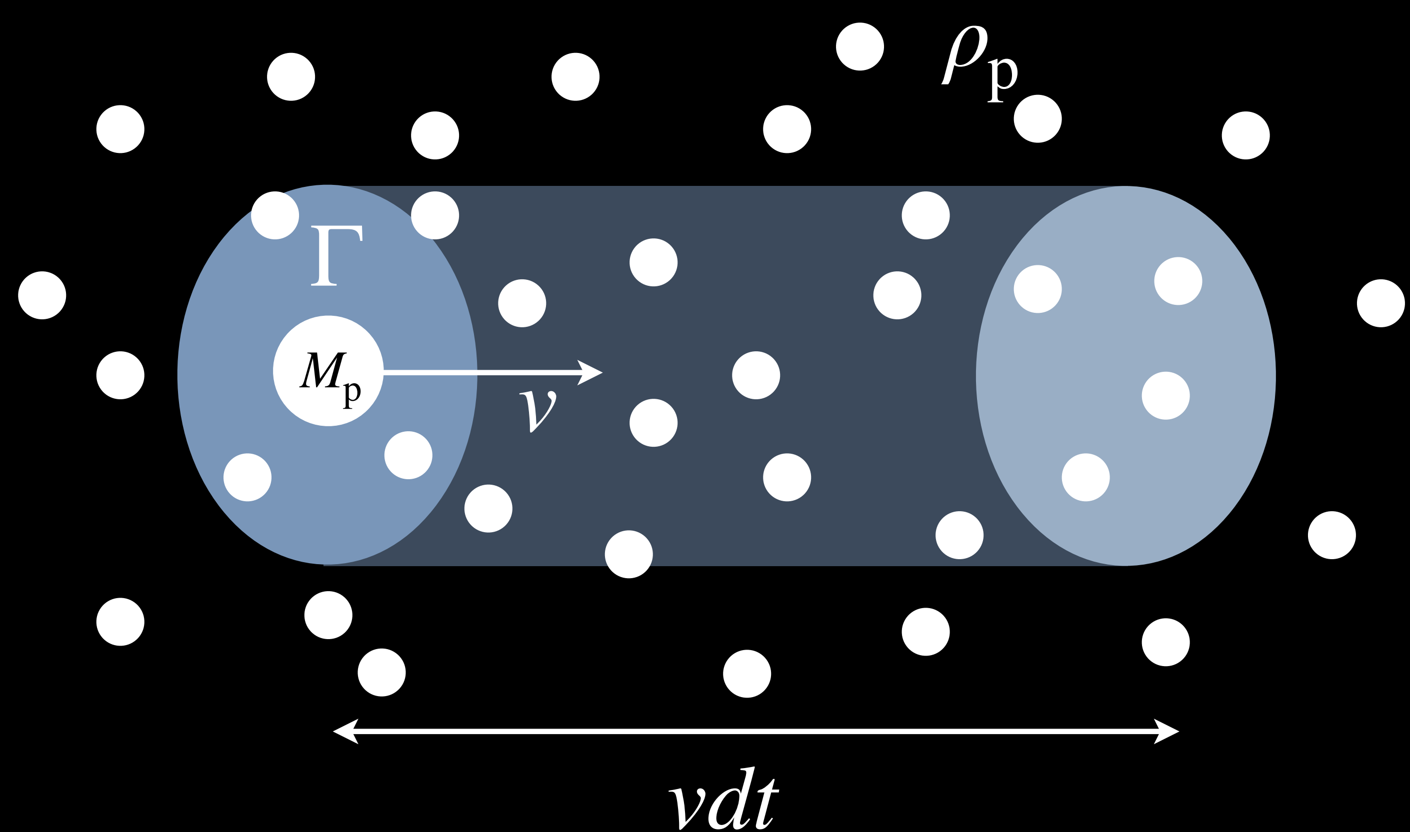
- The two-body approximation fails in the limit of low random velocities. This comes about because the encounter timescale becomes non-negligible compared to the orbital timescale.
    - Dispersion dominated regime: $\sigma > v_{\mathrm{H}}$ (2 body problem)
    - Shear dominated regime: $\sigma < v_{\mathrm{H}}$ (3 body problem)
- Planetesimal ejection is possible if planet escape velocity is greater than the system escape velocity:

$$\frac{v_{\mathrm{esc,p}}}{v_{\mathrm{esc},*}} \approx 0.15 \left(\frac{m}{M_{\oplus}}\right)^{1/3} \left(\frac{a}{\mathrm{au}}\right)^{1/2} \left(\frac{M_*}{M_{\odot}}\right)^{-1/2}$$

    - Massive planets further out can eject planetesimals
    - Growth is easier in the inner regions

GROWTH RATE

- One big body accreting from a background of smaller bodies:



$$dM_p = \rho_p \Gamma v dt$$

$$\rho_p = \frac{\Sigma_p}{2H_p}$$

$$\frac{H_p}{a_p} \approx \frac{v}{v_K} = \frac{v}{a_p \Omega_K}$$

- For an isotropic velocity distribution: $$\frac{dM_p}{dt} = \frac{\sqrt{3}}{2} \Sigma_p \Omega_K \Gamma$$

- For uniform intrinsic density ρ_{int} and constant Γ:

$$\frac{dR_s}{dt} = \frac{\sqrt{3}}{8} \frac{\Sigma_p \Omega_K}{\rho_{int}} \Gamma \approx 1 \frac{\mathrm{cm}}{\mathrm{yr}} \cdot \Gamma \left(\frac{\Sigma_p}{10 \frac{\mathrm{g}}{\mathrm{cm}^2}} \right)$$

10 km body would need
$\gtrsim$10 Myrs to grow!
→ Γ needs to be very large!

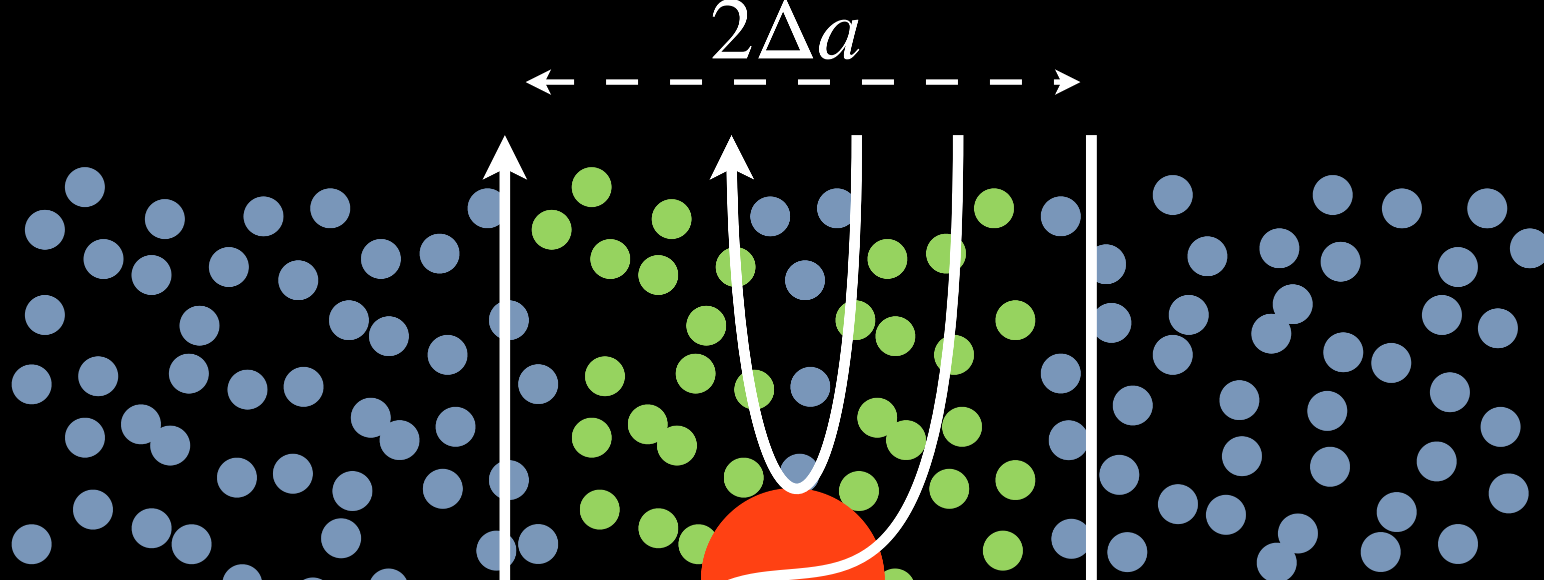
GROWTH RATE

- If $v_{\rm esc} \gg \sigma$, then:

$$\Gamma \approx \Gamma_{\rm geo} \frac{2GM_{\rm p}}{\sigma^2 R_{\rm s}} \qquad \frac{dM_{\rm p}}{dt} = \frac{\pi\sqrt{3}}{2}\Sigma_{\rm p}\Omega_{\rm K}R_{\rm s}\frac{GM_{\rm p}}{\sigma^2} \propto M_{\rm p}^{4/3} \propto R_{\rm s}^4$$

  - The bigger the mass, the faster it grows. Naively integrating this gives infinite masses in finite times! In reality, a massive body will begin to stir the environment and increase σ.

- In the shear regime, the feeding zone is $\Delta a \approx 2.3 r_{\rm H}$



GROWTH RATE

- Average velocity from shear at $\pm 0.75\Delta a$:

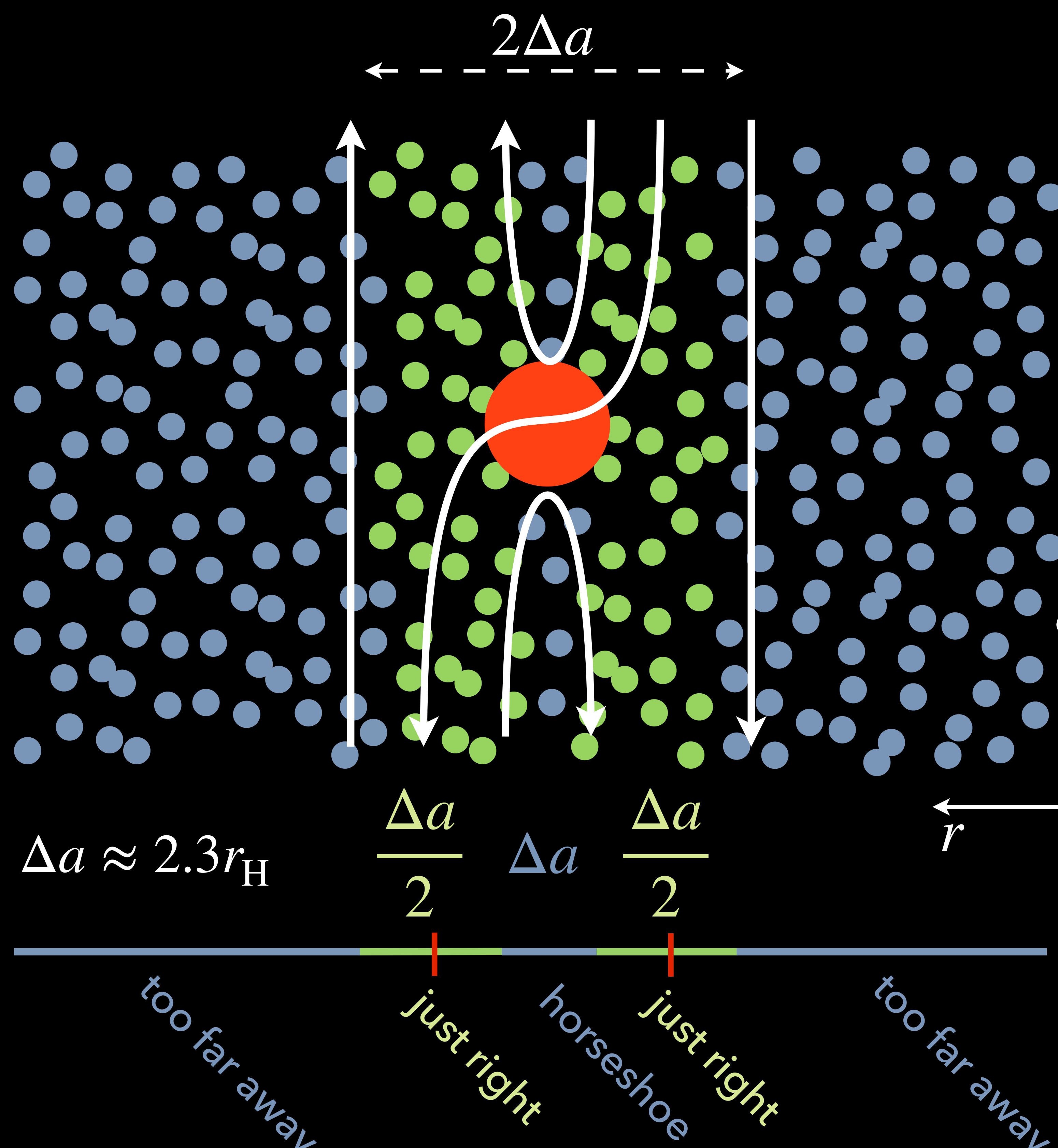
$$\Delta v = 0.75\Delta a \cdot a \cdot \left| \frac{d\Omega_K}{da} \right|$$

$$= \frac{9}{8}\Delta a \Omega_K$$

- The mass flow into the Hill sphere:

$$\frac{dM_H}{dt} = \Sigma_p \Delta a \Delta v$$

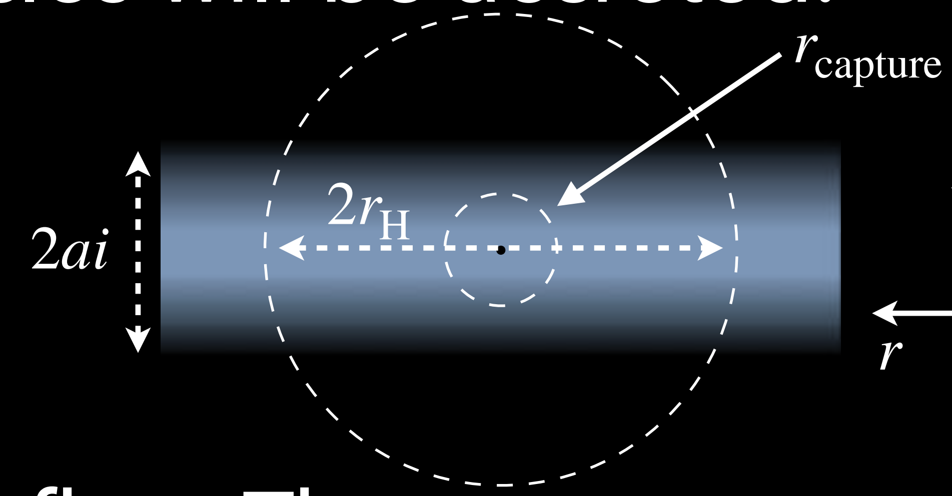
$$= \frac{9}{8}\Delta a^2 \Omega_K \Sigma_p$$



GROWTH RATE

- In the vertical direction, the planetesimal scale height is important and only a fraction of particles will be accreted.

$$f = \frac{\text{capture cross section}}{\text{captured fraction}} = \begin{cases} \frac{\pi r_\text{capture}^2}{2r_\text{H} \cdot 2ai} & \text{if } ai > r_\text{capture} \\ \frac{r_\text{capture}}{r_\text{H}} & \text{otherwise} \end{cases}$$



- In a cold thin disc, we get a 2D planar flow. The rate at which planetesimals enter the Hill sphere remains unaltered, but the fraction of planetesimals accreted is reduced.

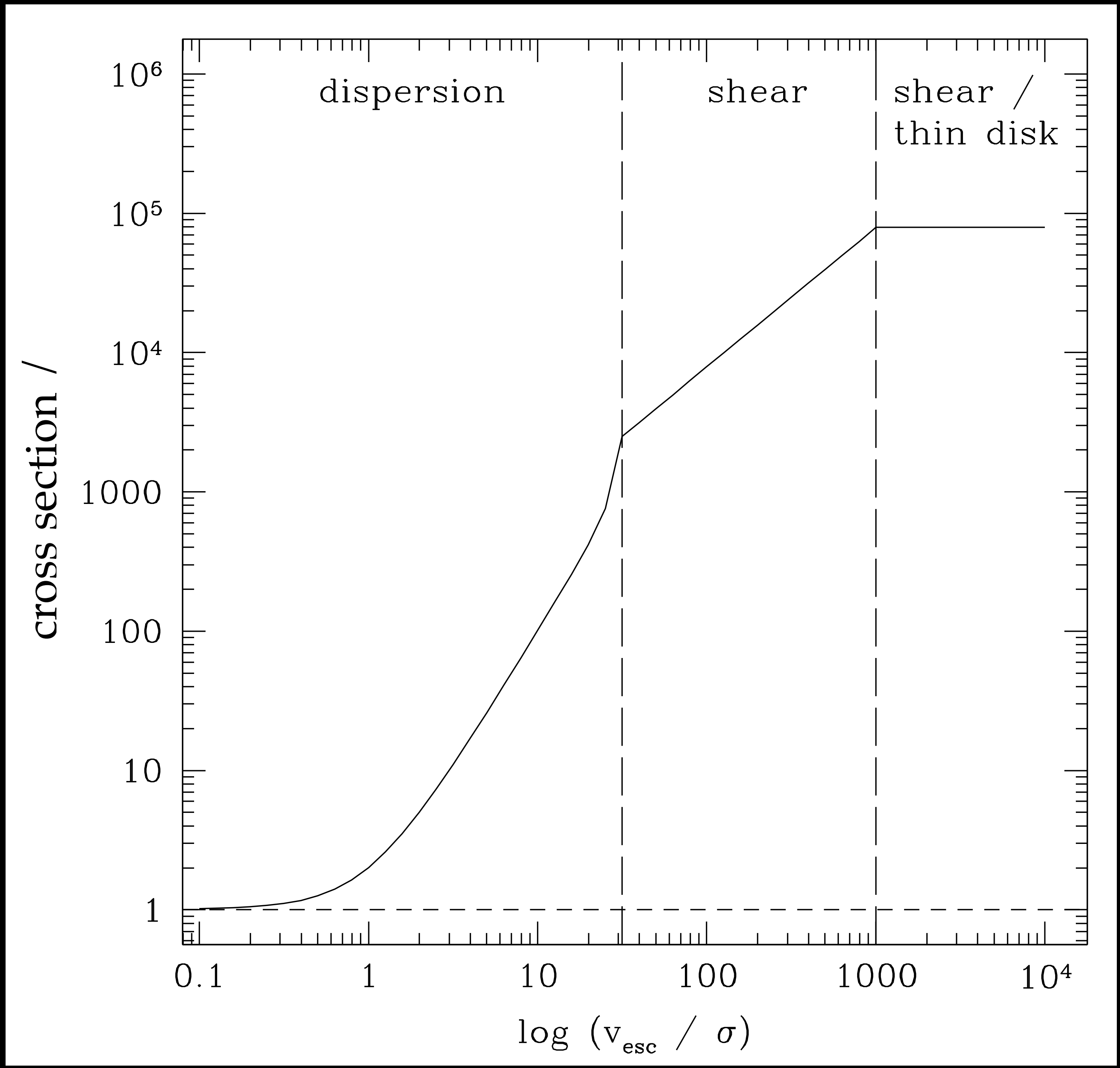
Mass dependence in the gravitational focusing term is partially cancelled

- Assuming $\Delta v \approx \Delta a \Omega_\text{K}$ and $ai > r_\text{capture}$:

$$\frac{dM_\text{p}}{dt} = \frac{9}{8}\Delta a^2 \Omega_\text{K} \Sigma_\text{p} f = \frac{9}{32}\frac{\Delta a^2}{a\, i\, r_\text{H}} \Sigma_\text{P} \Omega_\text{K} \pi R_\text{s}^2 \left[1 + \frac{v_\text{esc}^2}{(\Delta a \Omega_\text{K})^2}\right]$$

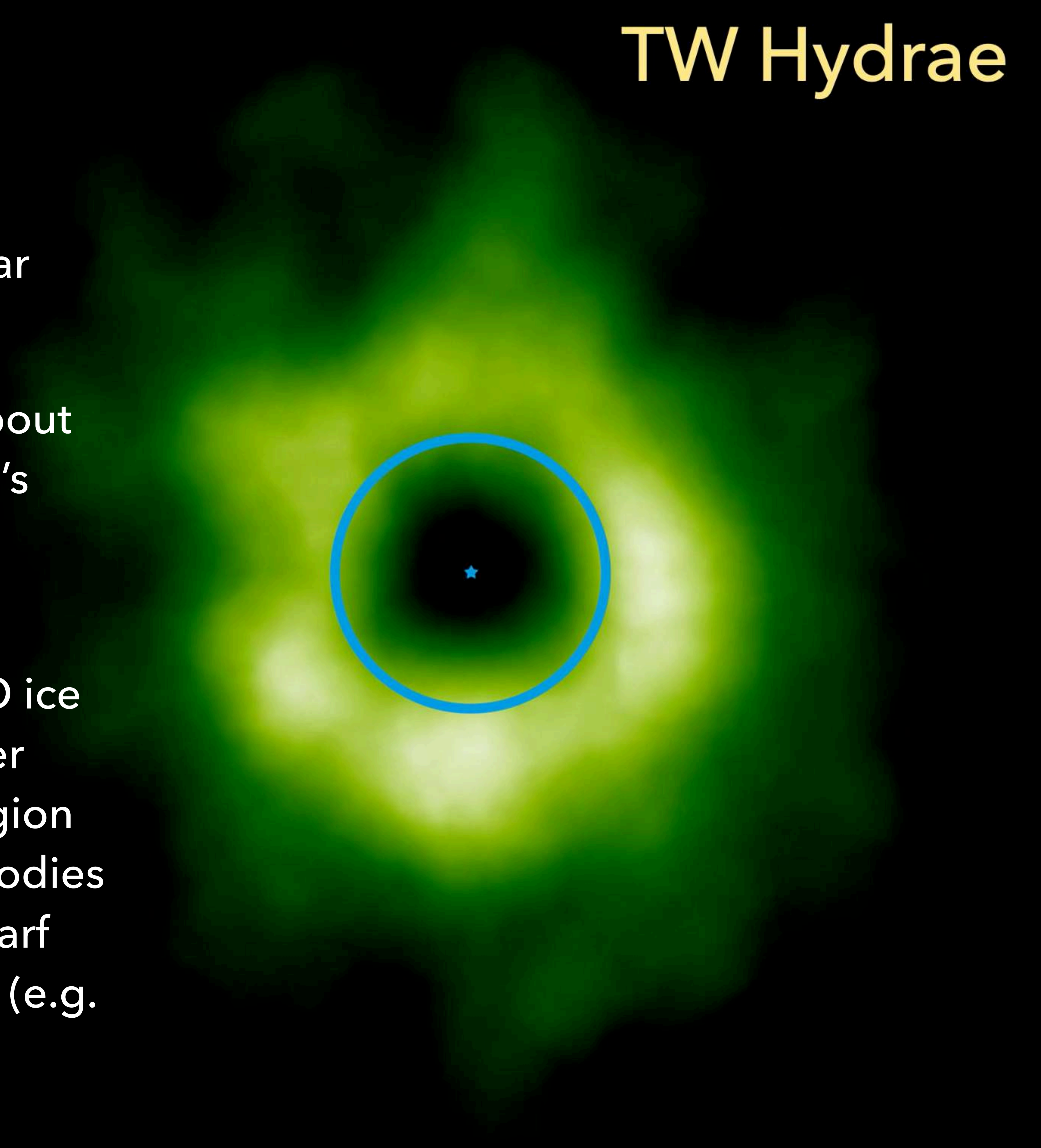
GROWTH RATE

- For $v_{esc}/\sigma < 1$, gravitational focusing is irrelevant and the cross-section is close to the geometric cross-section.
- For $v_{esc}/\sigma > 1$, gravitational focusing becomes important. It is still dominated by dispersion, but the cross-section increases quadratically.
- In the shear dominated regime, the increase in cross-section is slowed.
- When the disc thickness falls below the scale of the capture radius, the effective cross-section is constant.



SNOW LINES

- ALMA image of CO snow around the star TW Hydrae.
- The blue circle is about the size of Neptune's orbit in our Solar System.
- The transition to CO ice could mark the inner boundary of the region where smaller icy bodies like comets and dwarf planets would form (e.g. Pluto and Eris).



SNOW LINES

- Find the radius where $T_{\rm mid}(R) = T_{\rm snow}$. We approximate $T_{\rm mid}$ using the blackbody emission from accretion and irradiation.
- The luminosity generated by accretion through the disc:

$$L_{\rm acc} = \sigma T_{\rm eff,acc}^4 = \frac{3}{8\pi}\frac{GM_*\dot{M}}{R^3}\left(1 - \sqrt{\frac{r_*}{R}}\right)$$

accounts for finite R_*

- Because the disc is optically thick, the temperature arising from the accretion luminosity is: ($\tau_{\rm R}$ is the Rosseland optical depth)

$$T_{\rm mid,acc}^4 = \frac{3}{4}\left(\tau_{\rm R} + \frac{2}{3}\right) T_{\rm eff,acc}^4$$

- Irradiation from the star is absorbed and remitted

$$F_{\rm disc} = F_{\rm irr} \longrightarrow \frac{\alpha}{2}\left(\frac{R_*}{a_P}\right)^2 T_*^4 = T_{\rm irr}^4$$

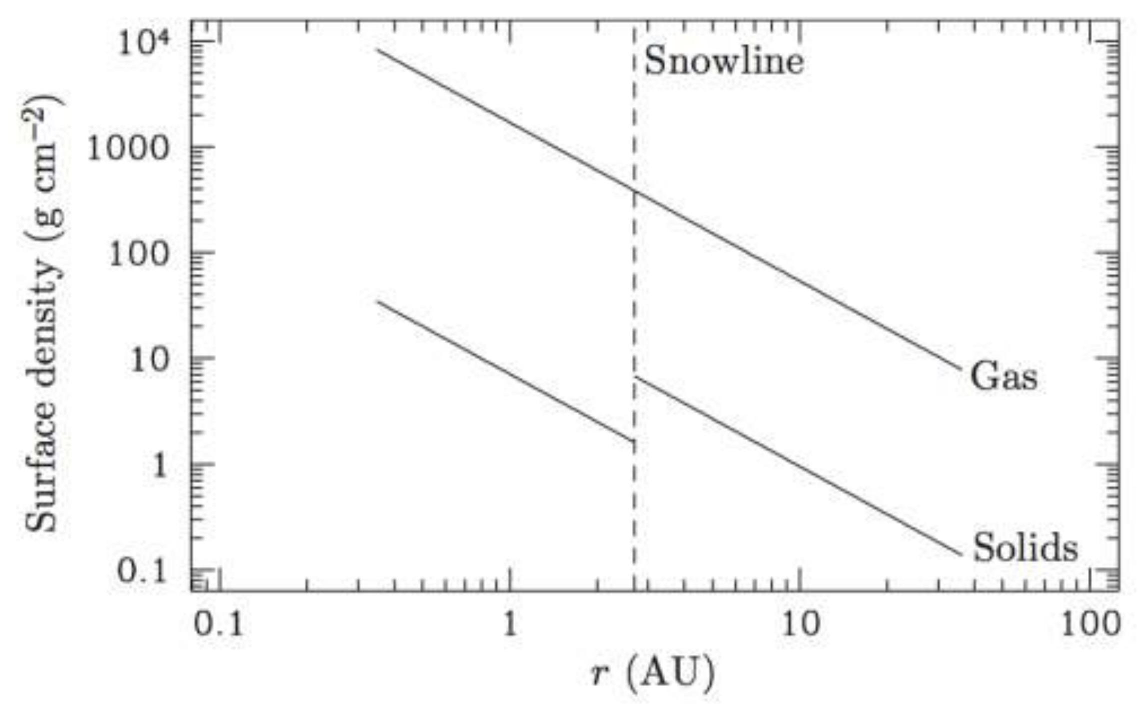
grazing angle

$$\alpha \approx 0.4\frac{R_*}{a_P}$$

- Combining these results gives: $T_{\rm mid}(R)^4 = T_{\rm mid,acc}^4 + T_{\rm irr}^4 \longrightarrow R_{\rm snow}$.

SNOW LINES

- For water ice: $T_{\text{snow}} \sim 150\text{–}170$ K, corresponding to $R \sim 1\text{–}3$ au. The snow line for the Solar System was probably at $R = 2.7$ au (since the outer asteroids are icy and the inner asteroids are largely devoid of water).
- At the snow line, the density of solid particles increases suddenly. This increase in solid-particle surface density affects the time-scales and mass-scales of planets that form beyond the snow line.



- Gas giants form more easily beyond the snow line, since cores that form beyond the snow line are more massive and have a longer time to accrete gas from the disk before it dissipates.

ISOLATION MASS

- The timescale for planet formation is roughly $\tau \propto 1/\Sigma$ so planetary cores which form beyond the snow-line are much larger than those that form within it.
- Isolation mass: maximum mass a body can achieve through planetesimal accretion ($M_{\rm iso} \propto \Sigma^{3/2} a_P^3$).
- Amplification of the solid surface density by a factor of ~3-4 at the snow line leads to an amplified isolation mass by a factor of ~5-8.
- The snow-line facilitates gas giant formation by helping cores to reach runaway gas accretion sooner. Timing is crucial because they must accrete the gas before the disc is dispersed.

SUMMARY 1/2

- Disc temperature is important for determining the condensation sequence, which affects the chemistry of solids in the disc.
    - CI-chondrites show the lease processing and closely match the abundances in the Sun. Give a good window on the chemical composition of the solar nebula.
- Growth of small grains initially occurs through collisions, but the growth efficiency drops near cm sizes due to bouncing, fragmentation, and radial drift.
    - Dust traps are essential to prevent the solid material from draining onto the star.
    - Likely need another mechanism to make the jump to planetesimal sizes.

SUMMARY 2/2

- Planetesimals again grow through collisions, but are now large enough for self-gravity to play an important role.
    - Gravitational focusing and internal structure.
- The growth rate of planetesimals is sensitive to the velocity dispersion. As planets form, the velocity dispersion will change (excited eccentricities and ejection).
- Once planets get too large, they reach an isolation mass, where the growth due to planetesimal accretion slows down dramatically.
    - Snow lines play an important role in accelerating core formation and allowing cores to reach the runaway gas accretion phase before the gas in the disc is dispersed.